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ON FULLY COMMUTATIVE (*n*,*m*)-GROUPOIDS

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A b s t r a c t: In several papers (for ex. [2] and [8]) on fully commutative vector valued groupoids, corresponding properties by means of commutative vector valued groupoids are given. The goal of this paper is to give a direct investigation of the class of fully commutative vector valued groupoids without leaving the very class.

Key words: vector valued operation, fully commutative vector valued groupoid (semigroup, group).

1. PRELIMINARIES

In this section we introduce some notations and define some notions used in this paper. We also give a short summary of sections 2, 3 and 4.

1.1. Throughout this paper, $Q^{(+)}$ denotes a free commutative semigroup with a given basis Q, where the operation is denoted multiplicatively, i.e., the operation sign is omitted.

Below we give a list of notations, notions, and properties used later on in the paper:

1) The mapping $x \mapsto |x|$ is the homomorphism from $Q^{(+)}$ into the additive semigroup of positive integers, such that $|b| = 1$ for each $b \in Q$.

2) If r is a positive integer, then

$$Q^{(r)} = \{x \in Q^{(+)}; \quad |x| = r\}.$$

3) For a given $a \in Q$, $x \mapsto |x|$, is the homomorphism from $Q^{(+)}$ into the additive semigroup of non-negative integers, such that $|a|_a = 1$, $|b|_a = 0$, for $b \neq a$.

4) For $x \in Q^{(+)}$ the set $cn(x) = \{b \in Q: |x|_b > 0\}$ is the *content* of *x*.

5) If $x, y \in Q^{(+)}$, then

$$\rho(x, y) = \sum_{a \in Q} \big| |x|_a - |y|_a \big|$$

is called the *distance between x and y*.

From the given definitions it follows that:

6) The semigroup $Q^{(+)}$ is cancellative.

7) For each $x \in Q^{(+)}$, $|x| = \sum_{a \in Q} |x|_a = \sum_{a \in cn(x)} |x|_a$ is a positive integer.

8) For each pair $x, y \in Q^{(+)}$, the distance $\rho(x, y)$ is a non-negative integer, where $\rho(x, y) = 0$ iff $x = y$.

For technical reasons, we will add to $Q^{(+)}$ an exterior unit e (i.e $e \notin Q^{(+)}$) and obtain that $Q^{(*)} = Q^{(+)} \cup \{e\}$ is a free commutative monoid with a unit element and the basis Q. Therefore e has the following properties:

9) $e \cdot x = x = x \cdot e$, for every $x \in Q^{(*)}$; $|e| = 0$, $cn(e) = \emptyset$, $Q^{(0)} = \{e\}$, $x^0 = e$, $\rho(e, x) = |x|$, for each $x \in Q^{(*)}$.

1.2. Further on we assume that (n, m) is a pair of positive integers, such that $n - m = k \geq 1$.

If $f: x \mapsto f(x)$ is a mapping from $Q^{(n)}$ into $Q^{(m)}$, then we call f a *fully commutative (n, m)-operation* and the pair $\mathbf{Q} = (Q, f)$ a *fully commutative (n, m) -groupoid*. This notion is introduced first in [3] and [2], but it seems necessary to explain the prefix "fully commutative". One of the reasons is the fact that the term "commutative (*n*, *m*)-groupoid" is used in [1] and [4] for a pair of the form (*Q*, *f*), such that *f* is a mapping from Q^n into Q^m with a corresponding property of commutativity.

In this paper we will consider only *fully commutative (n, m)-groupoids*, and thus we will usually omit the prefix "fully commutative".

1.3. In section 2 we define an infinite set $\{f^r : r \geq 2\}$ of polynomial mappings in an (n, m)-groupoid (Q, f), and here we will concentrate only to f^2. Namely, f^2 is a mapping from $Q^{(n)} \times Q^{(k)}$ into $Q^{(m)}$ defined by

$$f^2(x, y) = f(f(x)y), \text{ for } x \in Q^{(n)}, y \in Q^{(k)}.$$

Using this, it follows that there is at most one $(n + k, m)$-operation $f^{(2)}$, such that $f^{(2)}(xy) = f^2(x, y)$. It comes out that for each pair (n, m) and a set Q with at least two distinct elements, there is an (n, m)-groupoid (Q, f) which does not allow $(n + k, m)$-operation $f^{(2)}$ with the mentioned property. However, if such an $(n + k, m)$-operation exists, then (Q, f) is said to be a *fully commutative* (n, m)-*semigroup*. This definition of (n, m)-semigroup is equivalent to the definition in [2].

It is assumed in section 3 that (Q, f) is a given $(m + k, m)$-semigroup and the General Associative Law is proved. We note that this result is also proved in [2]. It took place in this paper, too, because its proof is such that we do not leave the class of fully commutative (n, m)-groupoids.

2. POLYNOMIAL MAPPINGS IN AN (n, m)-GROUPOID

First we will prove the following:

Proposition 2.1. *If Q has at least two distinct elements, then there is an (n, m)-groupoid* (Q, f) such that

$$x\,y = x'y' \;\&\; f(f(x)\,y) \neq f(f(x')\,y'), \tag{2.1}$$

for some $x,\ x' \in Q^{(n)},\ \ y, y' \in Q^{(k)}$.

Proof. Let $a,\ b \in Q,\ a \neq b$ and let $x, x' \in Q^{(n)},\ \ y, y' \in Q^{(k)}$ be defined as follows:

$$x = a^n, \quad y = b^k, \quad x' = a^{n-1}b, \quad y' = ab^{k-1}.$$

If f is a mapping from $Q^{(n)}$ into $Q^{(m)}$ with the properties

$$a^m = f(a^n) = f(a^m b^k),\ b^m = f(a^{n-1}b) = f(ab^{n-1}),$$

then $x, x' \in Q^{(n)}$ and $y, y' \in Q^{(k)}$ are such that (2.1) holds. □

Below we assume that (Q, f) is an (n, m)-groupoid.

We define a set of mappings $\{f^r : r \geq 1\}$ as follows. First, $f^1 = f$. Assuming that $f^r : D_r \to Q^{(m)}$ is defined, we define $f^{r+1} : D_{r+1} \to Q^{(m)}$, where $D_{r+1} = D_r \times Q^{(k)}$ $(D_1 = Q^{(n)})$, by

$$f^{r+1}(x, y) = f(f^r(x)y), \tag{2.2}$$

for $x \in D_r$, $y \in Q^{(k)}$.

Since $D_1 = Q^{(n)}$, one obtains that

$$D_r = Q^{(n)} \times (Q^{(k)})^{r-1} = \{(x_1, x_2, \ldots, x_r) : x_1 \in Q^{(n)}, x_2, \ldots, x_r \in Q^{(k)}\},$$

for each $r \geq 2$.

The following proposition will be used in section 3.

Proposition 2.2. *For each pair (p, q) of positive integers and* $x_1 \in Q^{(n)}$, $x_2, \ldots, x_{p+q} \in Q^{(k)}$, *it holds that*

$$f^{p+q}(x_1, x_2, \ldots, x_{p+q}) = f^p(f^q(x_1, \ldots, x_q)x_{q+1}, \ldots, x_{p+q}). \tag{2.3}$$

Proof. For $p = 1$, (2.3) holds by (2.2). For $q = 1$, (2.3) can be shown by induction on p, and for $p \geq 2$, $q \geq 2$ by induction on $p + q$. □

The following statement is clear.

Proposition 2.3. *If* $r \geq 2$, *then there is at most one* $(m + rk, m)$*-operation* $f^{(r)}$, *such that*

$$f^{(r)}(x_1 x_2 \ldots x_r) = f^r(x_1, x_2, \ldots, x_r) \tag{2.4}$$

for each $(x_1, x_2, \ldots, x_r) \in D_r$. *Such an operation* $f^{(r)}$ *do exist iff the equality*:

$$f^r(x_1, x_2, \ldots, x_r) = f^r(x'_1, x'_2, \ldots, x'_r), \tag{2.5}$$

holds for every $(x_1, x_2, \ldots, x_r), (x'_1, x'_2, \ldots, x'_r) \in D_r$, *such that* $x_1 x_2 \ldots x_r = x'_1 x'_2 \ldots x'_r$. □

We say that (*Q, f*) is a *fully commutative* (*n, m*)*-semigroup iff* $(Q, f^{(2)})$ is a well-defined fully commutative (*n* + *k, m*)-groupoid. As is mentioned in 1.2, we will omit the prefix "fully commutative" and say simply "(*n* ,*m*)-semigroup". By Proposition 2.3, it follows that (*Q, f*) is an (*n, m*)-semigroup iff the equality

$$f^2(x, y) = f^2(x', y'), \tag{2.6}$$

i.e.,

$$f(f(x)y) = f(f(x')y'), \tag{2.7}$$

holds for all $x, x' \in Q^{(n)}$, $y, y' \in Q^{(k)}$, such that $xy = x'y'$.

By the proposition that will be proved below, it follows that this definition of the notion of an (*n, m*)-semigroup is equivalent to the definition in [2].

Proposition 2.4. *The following conditions are equivalent:*

(*i*) (*Q, f*) *is an* (*n, m*)*-semigroup.*

(*ii*) *The equality*

$$f(f(xa)by) = f(f(xb)ay) \tag{2.8}$$

holds, for every $a, b \in Q$, $x \in Q^{(n-1)}$, $y \in Q^{(k-1)}$.

Proof. $(i) \Rightarrow (ii)$ follows by (2.7). Suppose that (*ii*) holds, but that (*i*) does not. Then there are $t, t' \in Q^{(n)}$ and $u, u' \in Q^{(k)}$, such that

$$tu = t'u', \quad f(f(t)u) \neq f(f(t')u'), \tag{2.9}$$

where $\rho(u, u')$ is the lowest possible value. Then $u \neq u'$, because for $u = u'$ we would have $t = t'$, which contradicts the inequality in (2.9). Therefore

$s=\rho(u,u')>0$. Since $|u|=|u'|$, one obtains that there are $a,b\in Q$, such that $|u|_a > |u'|_a$, $|u|_b<|u'|_b$. Then there are $x\in Q^{(n-1)}$, $y\in Q^{(k-1)}$, such that $u = ay$, $t=xb$. By (*ii*):

$$f(f(t)u)=f(f(xb)ay)=f(f(xa)by),\quad r=\rho\,(by,u')<s=\rho\,(u,u'),$$

which is a contradiction of the minimality of s. □

Remark. Supposing that $k\geq m$, beside the infinite set of polynomial mappings $\{f^r : r\geq 1\}$ in the given (*n, m*)-groupoid (*Q, f*), one can also define a finite set of polynomials as follows.

First, since $k\geq m$, there is a unique pair of integers (q, p) such that $k=qm+p$, $q\geq 1$, $0\leq p<m$. Therefore, if $x_1,x_2,\dots,x_r\in Q^{(n)}$, $y\in Q^{(s)}$ are such that $rm+s=n\,(=(q+1)m+p)$, then

$$z=f(f(x_1)f(x_2)\dots f(x_r)y)\in\; Q^{(m)} \tag{2.10}$$

and z does not depend on the disposition of $x_1,x_2,\dots,x_r$. For r = 1, we have $f^2(x_1,y)$, and hence we suppose that $r\geq 2$. Then, $s=(q+1-r)m+p$.

Let $(Q^{(n)})^{(+)}$ be a free commutative semigroup with the basis $Q^{(n)}$, where the product of $x_1,x_2,\dots,x_r\in Q^{(n)}$ will be denoted by $x_1\bullet x_2\bullet\cdots\bullet x_r$ in $(Q^{(n)})^{(+)}$ to make a distinction of their product $x_1x_2\dots x_r$ in $Q^{(+)}$. If we put $D_{[r]}=(Q^{(n)})^{(r)}\times Q^{(s)}$ and

$$f^{[r+1]}(x_1\bullet x_2\bullet\cdots\bullet x_r,y)=f(f(x_1)f(x_2)\dots f(x_r)y), \tag{2.11}$$

we come to a new set $\{f^{[3]},f^{[4]},\dots,f^{[q+2]}\}$ of polynomial operations, such that $f^{[r+1]}$ has the domain $D_{[r]}$ and is defined by (2.11). Also, for $r=q+1$ in (2.11) we have that $s=p$. Hence, for $p=0$ and $r=q+1$ we have that $D_{[r]}=(Q^{(n)})^{(d+1)}$, and (2.11) obtains the form

$$f^{[q+1]}(x_1\bullet x_2\bullet\dots\bullet x_{q+1})=f(f(x_1)f(x_2)\dots f(x_{q+1})).$$

3. THE GENERAL ASSOCIATIVE LAW

Below we assume that (Q, f) is an (n, m)-semigroup. We will prove the following proposition in several steps.

Theorem 3.1. *For every positive integer r,* $(Q, f^{(r)})$ *is an* $(m + rk, m)$-*semigroup.*

(For $r = 1$, the theorem is true by assumption.)

First, we will prove the following:

Lemma 3.2. *Let* $r \geq 2$, $x_1 \in Q^{(n)}$ *and* $x_\nu \in Q^{(k)}$ *for each* $\nu : 2 \leq \nu \leq r$, *and let* $x_i = y_i a$, $x_j = b y_j$, $a, b \in Q$, *for given* i, j, *such that* $1 \leq i < j \leq r$. Then

$$f^r(x_1, x_2, \ldots, x_r) = f^r(x_1', x_2', \ldots, x_r'), \tag{3.1}$$

where $x_\nu' = x_\nu$ *for* $\nu \neq i, j$, *and* $x_i' = y_i b$, $x_j' = a y_j$.

Proof. It suffices only to prove the case $j = i + 1$.

For $r = 2$, (3.1) is in fact (2.8).

Suppose that (3.1) is true for a given $r \geq 2$. To show that (3.1) is also true when r is changed to $r + 1$, we will use the equality (2.3) for $p + q = r + 1$. If $i < r$, i.e. $i + 1 < r + 1$, then we put $q = i + 1$ in (2.3) and use the inductive hypothesis. For $i = r$, we put $q = 1$ in (2.3) and we again use the inductive hypothesis. □

Using this lemma in almost the same way as the proof of the part $(ii) \Rightarrow (i)$ in Proposition 2.4, one can show the following:

Proposition 3.3. *For each positive integer r,* $(Q, f^{(r)})$ *is an* $(m + rk, m)$-*groupoid.* □

It remains to be shown that, for each $r \geq 2$, $(Q, f^{(r)})$ is an $(m + rk, m)$-semigroup, i.e., that

$$f^{(r)}(f^{(r)}(xa)by) = f^{(r)}(f^{(r)}(xb)ay) \tag{3.2}$$

holds for every $a, b \in Q$, $x, y \in Q^{(+)}$, such that $|xa| = m + rk$, $|by| = rk$.

Therefore, representing xa and by in the form

$$xa = x_1 x_2 \cdots x_r a, \ by = by_1 y_2 \cdots y_r,$$

where $|x_1| = n, |x_2| = \cdots = |x_{r-1}| = |x_r a| = k = |by_1| = |y_2| = \cdots = |y_r|$, we use (2.4) and (2.3) to represent the left-hand side of (3.2) in the form

$$f^{2r}(x_1, x_2, \ldots, x_{r-1}, x_r a, by_1, y_2, \ldots, y_r). \tag{3.3}$$

Then, applying equality (3.1) in (3.3), and performing a reverse procedure from the previous one, we would get the right-hand side of (3.2).

This completes the proof of Theorem 3.1.

The equality in the following proposition is known as the *General Associative Law* (GAL) for fully commutative (n, m)-semigroups.

Proposition 3.4. *Let* (Q, f) *be an* (n, m)*-semigroup. Then*

$$f^{(r)}(f^{(p_1)}(x_1) f^{(p_2)}(x_2) \cdots f^{(p_s)}(x_s)\, y) = f^{(r+p_1+p_2+\cdots+p_s)}(x_1 x_2 \cdots x_s\, y), \tag{3.4}$$

where $|x_\nu| = m + kp_\nu$ *for* $1 \le \nu \le s$, $sm + |y| = m + rk$, *and* r, p, s *are given positive integers. (Here, it is possible that* $|y| = 0$, *i.e.* $y = e$*).*

Proof. For $s = 1$, (3.4) has the form

$$f^{(r)}(f^{(p)}(x) y) = f^{(r+p)}(xy) \tag{3.5}$$

(where p stands for p_1, and x for x_1) which can be proved using (2.4) and (2.3). Then, supposing that (3.4) is true for some $s - 1 \ge l$, we substitute $f^{(p_1)}(x_1) y$ with z in the left-hand side of (3.4), apply first the inductive hypothesis and then (3.5), we obtain (3.4). □

Remark 1. Fully commutative $(m + k, m)$-groups are defined in [2]. Namely, a fully commutative $(m + k$, m)-semigroup is a *fully commutative* $(m + k, m)$*-group* if for each $a \in Q^{(k)}$, $b \in Q^{(m)}$ there is an $x \in Q^{(m)}$ such that $f(ax) = b$. These are also considered in: [7], [8], [9], [6]. In [2] it is shown that every infinite set is a carrier of a fully commutative $(m + k, m)$-group, that for

$m \geq 2$ there is no finite fully commutative $(m + k, m)$-group with more than two elements, and that there are non-isomorphic fully commutative $(m + k, m)$-groups with a two-element carrier. In [7] it is noted that a structure of a fully commutative $(m + k, m)$-group can be built on every algebraically closed field, and in [8] the class of affine and the class of projective $(m + k, m)$-groups are defined. In [9], the class of affine and the class of projective $(m + k, m)$-groups with sets of complex numbers as their carriers are studied. We also note that in [5], [8] and [10] several axiom systems of the class of $(m + k, m)$-groups are obtained.

Following the idea of [10], a proposition analogous to Theorem 1 in [5] for fully commutative $(m + k, m)$-groups can be stated and proved.

Remark 2. The class of fully commutative $(m + k, m)$-groups is a subclass of the class of fully commutative $(m + k, m)$-quasigroups, defined in [3]. Namely, a fully commutative $(m + k, m)$-groupoid (Q, f) is said to be a $(m + k, m)$-quasigroup iff for each $x \in Q^{(k)}$, $y \in Q^{(m)}$ there is a unique $z \in Q^{(m)}$, such that $f(xz) = y$. It is shown in [3] that each cancellative fully commutative $(m + k, m)$-groupoid is embeddable in a fully commutative $(m + k, m)$-quasigroup. It is also shown there that, if $q \geq 3$, then there is a fully commutative $(q, q-1)$-quasigroup with $q + 1$ elements. On the other hand, there is no fully commutative $(m + k, m)$-quasigroup with $q + 1$ elements for $2 \leq q \leq m$. We note that there are many open problems on finite fully commutative $(m + k, m)$-quasigroups.

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Р е з и м е

ЗА ПОТПОЛНО КОМУТАТИВНИТЕ (*n*, *m*)-ГРУПОИДИ

Во неколку трудови (на пример [2] и [8]) за потполно комутативни векторско вредносни групоиди се разгледуваат соодветни својства со помош на комутативни векторско вредносни групоиди. Главната цел на оваа работа е „автономно“ испитување на класата потполно комутативни векторско вредносни групоиди, т.е. испитување без да се напушти самата класа.

Клучни зборови: векторско вредносна оперција, потполно комутативен векторско вредносен групоид (полугрупа, група)

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