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FREE GROUPOIDS DEFINED BY THE IDENTITY $(xy)y = yx$

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Abstract. A construction of free groupoids in the variety $\mathcal{V}$ defined by Stein identity $(xy)y = yx$ [1] is given. The construction is canonical i.e. the free groupoid is constructed with universe a subset of the absolutely free groupoid and suitably defined operation. In fact a countable closed set of identities that are consequences of the defining one is obtained.

Stein identities $xy \cdot y = yx$ and $x \cdot xy = yx$, which are dual to each other, are interesting from many aspects. For each quasigroup satisfying any of the stated identities there is an orthogonal mate which is a quasigroup as well [1]. Stein identity together with $xy \cdot yx = x$ ($x \cdot xy = y$) define a class of cancellative groupoids with the following property: every groupoid generated by two elements has exactly four (five) elements [5].

In what follows $\mathcal{V}$ will denote the variety of groupoids satisfying the identity $xy \cdot y = yx$.

PROPOSITION 1. *Let $(G, \cdot)$ be a $\mathcal{V}$-groupoid. Then for each $x, y \in G$*
(i) $x \cdot yx = xy \cdot x$;
(ii) $x^2 \cdot x^2 = x^2$.

Proof. *(i)* $x \cdot yx = (yx \cdot x)x = xy \cdot x$;
(ii) $x^2 \cdot x^2 = (x^2 \cdot x) \cdot x^2 = (x \cdot x^2) \cdot x^2 = x^2 \cdot x = x^2$. ∎

Let $B \neq \emptyset$ and let $\boldsymbol{T}_B = (T_B, \cdot)$ denote the absolutely free groupoid with a free base B. Define length $|u|$ of a term $u \in T_B$ inductively by

$$u \in B \implies |u| = 1, \qquad u = vw \implies |u| = |v| + |w|.$$

By $t(x)$ we denote a term t in which only one variable x appears. Also $t(u)$ will denote the term obtained from $t(x)$ by replacing each occurrence of x by the term u. Further on $|u|_t$ will denote the number of occurrences of u in t. The next statement is a consequence of Proposition 1 *(i)*, *(ii)* and the defining identity.

COROLLARY 1. *If $t = t(u)$ and $t \neq u$, then $t = u^2$ is true in $\mathcal{V}$.* ∎

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We also need to define some notions that will abbreviate the denotation of some special forms of products. The definition is inductive.

$$x^{\langle 0\rangle}y = y,\ x^{\langle 1\rangle}y = xy,\ x^{\langle n+1\rangle}y = x\cdot(x^{\langle n\rangle}y)$$

Let $S = \{u \in T_B |\ u$ does not contain a subterm of the forms (1)–(5)$\}$.

(1) $t(u),\ u \in T_B,\ |u|_{t(u)} > 2$

(2) $((uv)^{\langle n\rangle}(vu))((uv)^{\langle n\rangle}u)$

(3) $((uv)^{\langle n\rangle}u)((uv)^{\langle n\rangle}(vu))$

(4) $((uv)^{\langle n+1\rangle}u)((uv)^{\langle n\rangle}(vu))$

(5) $((uv)^{\langle n\rangle}(vu))((uv)^{\langle n+1\rangle}u)$

where $n \geq 0,\ u, v \in T_B,\ u \neq v$. For $t, s \in S$ let

$$t * s := \begin{cases} ts, & ts \in S \\ u^2, & t = t(u),\ s = s(u) \\ uv, & ts = ((uv)^{\langle n\rangle}(vu))((uv)^{\langle n\rangle}u) \\ (uv)^{\langle n+1\rangle}\cdot u, & ts = ((uv)^{\langle n\rangle}u)((uv)^{\langle n\rangle}(vu)) \\ uv, & ts = ((uv)^{\langle n+1\rangle}u)((uv)^{\langle n\rangle}(vu)) \\ (uv)^{\langle n+1\rangle}\cdot vu, & ts = ((uv)^{\langle n\rangle}(vu))((uv)^{\langle n+1\rangle}u) \end{cases}$$

where $n \geq 0,\ u, v \in T_B,\ u \neq v$.

We intend to prove that $(S, *)$ is a free groupoid in $\mathcal{V}$. The form (1) is a result of Corollary 1. For (2)–(5), as well as the corresponding products, one can find justification in the following property.

PROPOSITION 2. *In any $\mathcal{V}$-groupoid $(G,\cdot)$ the following hold:*

(a) $((uv)^{\langle n\rangle}(vu))((uv)^{\langle n\rangle}u) = uv,\ n \geq 0,\ u, v \in G,$

(b) $((uv)^{\langle n\rangle}u)((uv)^{\langle n\rangle}(vu)) = (uv)^{\langle n+1\rangle}\cdot u,\ n \geq 0,\ u, v \in G,$

(c) $((uv)^{\langle n+1\rangle}u)((uv)^{\langle n\rangle}(vu)) = uv,\ n \geq 0,\ u, v \in G,$

(d) $((uv)^{\langle n\rangle}(vu))((uv)^{\langle n+1\rangle}u) = (uv)^{\langle n+1\rangle}\cdot vu,\ n \geq 0,\ u, v \in G.$

Proof. We prove (a), (b), (c) and (d) simultaneously by induction on n. For $n = 0$, (a) is the identity $vu\cdot u = uv$; (b) is the statement (i) of Proposition 1; for (c) we use Proposition 1 (i) and the defining identity and get $(uv\cdot u)\cdot vu = (u\cdot vu)\cdot vu = (vu)u = uv$; and by similar transformations $vu\cdot(uv\cdot u) = ((uv\cdot u)\cdot vu)\cdot vu = ((u\cdot vu)\cdot vu)\cdot vu = (vu\cdot u)\cdot vu = uv\cdot vu$, i.e. (d) holds.

Assume that (a), (b), (c) and (d) hold in $(G,\cdot)$ for n. Then $((uv)^{\langle n+1\rangle}(vu))((uv)^{\langle n+1\rangle}u) \overset{(d),n}{=} (((uv)^{\langle n\rangle}(vu))\cdot((uv)^{\langle n+1\rangle}u)))\cdot, ((uv)^{\langle n+1\rangle}u) = ((uv)^{\langle n+1\rangle}u)\cdot((uv)^{\langle n\rangle}(vu)) \overset{(c),n}{=} uv$, hence (a) holds.

Also $((uv)^{\langle n+1\rangle}u)((uv)^{\langle n+1\rangle}(vu)) = (((uv)^{\langle n+1\rangle}(vu))\cdot((uv)^{\langle n+1\rangle}u))\cdot, ((uv)^{\langle n+1\rangle}u) \overset{(a),n+1}{=} uv\cdot((uv)^{\langle n+1\rangle}u) = (uv)^{\langle n+2\rangle}u$, so (b) holds.

We obtain (c) in the following way $((uv)^{\langle n+2\rangle}u)((uv)^{\langle n+1\rangle}(vu)) \overset{(b),n+1}{=} (((uv)^{\langle n+1\rangle}u)\cdot((uv)^{\langle n+1\rangle}(vu)))\cdot((uv)^{\langle n+1\rangle}(vu)) = ((uv)^{\langle n+1\rangle}(vu))\cdot((uv)^{\langle n+1\rangle}u) \overset{(a),n+1}{=} uv$.

Finally, (d) holds since
$((uv)^{\langle n+1\rangle}(vu))((uv)^{\langle n+2\rangle}u) = (((uv)^{\langle n+2\rangle}u)\cdot((uv)^{\langle n+1\rangle}(vu)))\cdot((uv)^{\langle n+1\rangle}(vu)) \overset{(c),n+1}{=} uv\cdot((uv)^{\langle n+1\rangle}(vu)) = (uv)^{\langle n+2\rangle}(vu)$. ∎

Note that if uv is of the form (2) or (4) then vu is of the form (3) or (5) respectively, and vice versa. Therefore, S has the following property.

PROPOSITION 3. *If* $uv \in S$ *then* $vu \in S$. ∎

Now we can prove the main result of this note.

THEOREM 1. $(S,*)$ *is a free groupoid in* $\mathcal{V}$ *with a free base* B.

Proof. By careful examination of the forms (1)–(5) it is clear that they are mutually exclusive. Let $t, s \in S$. If ts is of form (1), then $t*s = u^2 \in S$. If ts is of form (2) or (4) then uv or vu is a subterm of t hence, by Proposition 3, $uv \in S$. Finally, if ts is of form (3) or (5) then $t*s = uv\cdot t$ by definition and as before $uv, t \in S$; by checking all the forms it is clear that $uv\cdot t \in S$ where $t = (uv)^{\langle n\rangle}u$ or $t = (uv)^{\langle n\rangle}(vu)$ respectively. Hence, $(S,*)$ is a groupoid such that $B \subseteq S$. Also, $(S,*)$ is generated by the set B since if $uv \in S$ then $uv = u*v$ and we can use induction on the length of terms.

Next we show that $(S,*) \in \mathcal{V}$. Let $x, y \in S$. We check whether $(x*y)*y = y*x$ by distinguishing six cases for the product $x*y$.

(i) $x*y = xy$: Then, by Proposition 3, $y*x = yx$ and $xy*y = yx$ since $(xy)y = ((yx)^{\langle 0\rangle}(xy))\cdot((yx)^{\langle 0\rangle}y)$.

(ii) $x*y = u^2$, $x = x(u)$, $y = y(u)$: $(x*y)*y = u^2*y(u) = u^2 = y*x$.

(iii) $x*y = uv$, $x = (uv)^{\langle n\rangle}\cdot(vu)$, $y = (uv)^{\langle n\rangle}\cdot u$: $(x*y)*y = uv*y = (uv)*(uv)^{\langle n\rangle}\cdot u = (uv)^{\langle n+1\rangle}\cdot u = y*x$.

(iv) $x*y = (uv)^{\langle n+1\rangle}\cdot u$, $x = (uv)^{\langle n\rangle}\cdot u$, $y = (uv)^{\langle n\rangle}\cdot(vu)$: $(x*y)*y = ((uv)^{\langle n+1\rangle}\cdot u)*((uv)^{\langle n\rangle}\cdot(vu)) = uv = y*x$.

(v) $x*y = uv$, $x = (uv)^{\langle n+1\rangle}\cdot u$, $y = (uv)^{\langle n\rangle}\cdot(vu)$: $(x*y)*y = uv*((uv)^{\langle n\rangle}\cdot(vu)) = (uv)^{\langle n+1\rangle}\cdot(vu) = y*x$.

(vi) $x*y = (uv)^{\langle n+1\rangle}\cdot(vu)$, $x = (uv)^{\langle n\rangle}\cdot(vu)$, $y = (uv)^{\langle n+1\rangle}\cdot u$: $(x*y)*y = ((uv)^{\langle n+1\rangle}\cdot(vu))*((uv)^{\langle n+1\rangle}\cdot u) = uv = y*x$.

Finally, let $(G,\cdot) \in \mathcal{V}$ and $f\colon B \to G$ a mapping. Then $\hat{f} : S \to G$, defined by $\hat{f}(b) = f(b)$ for $b \in B$ and $\hat{f}(uv) = \hat{f}(u)\circ\hat{f}(v)$ for $uv \in S\setminus B$, is the desired homomorphism, by Proposition 2. ∎

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