Математички Билтен
Vol. 38(LXIV) No. 1
2014 (5-10)
Скопје, Македонија

ISSN 0351-336X
UDC: 515.124:515.122.55

ON CONTINUITY OF A $(3,1,\rho)-$METRIC

Sonja Čalamani[1], Dončo Dimovski[2]

Abstract. A given $(3,1,\rho)-$metric *d* on a set *M*, induces more than one topology τ on *M*. In general the map *d* from the third power of (M,τ) to the real numbers with the usual topology is not continuous. In this paper we consider one of the topologies τ on *M* and some additional conditions that will imply the continuity of *d*.

1. INTRODUCTION

The geometric properties, their axiomatic classification and the generalization of metric spaces have been considered in a lot of papers. We will mention some of them: K. Menger ([11]), V. Nemytzki, P.S. Aleksandrov ([13], [1]), Z. Mamuzic ([10]), S. Gähler ([8]), A. V. Arhangelskii, M. Choban, S. Nedev ([2], [3], [14]), R. Kopperman ([9]), J. Usan ([15]), B. C. Dhage, Z. Mustafa, B. Sims ([5], [12]). The notion of $(n,m,\rho)-$metric is introduced in [6]. Connections between some of the topologies induced by a $(3,1,\rho)-$metric d and topologies induced by a pseudo-o-metric, o-metric and symmetric are given in [7]. For a given $(3,1,\rho)-$metric d on set M, $j\in\{1,2\}$, seven topologies $\tau(G,d)$, $\tau(H,d)$, $\tau(D,d)$, $\tau(N,d)$, $\tau(W,d)$, $\tau(S,d)$ and $\tau(K,d)$ on M, induced by d, are defined in [4], and several properties of these topologies are shown.

In this paper we consider only the topology (G,d) induced by a $(3,1,\rho)-$metric d. For $\tau=\tau(G,d)$, we will state two conditions for d and show that these conditions imply the continuity of d, as a map from the third power of the topological space (M,τ) to the real numbers with the usual topology.

We recall the basic notions.

Let M be a nonempty set, and let $d:M^3\to R_0^+=[0,\infty)$. We state three conditions for such a map.

$(\boldsymbol{M}0)$ $d(x,x,x)=0,$ for any $x\in M$;

$(\boldsymbol{P})$ $d(x,y,z)=d(x,z,y)=d(y,x,z)$ for any $x,y,z\in M$; and

$(\boldsymbol{M}1)$ $d(x,y,z)d(x,y,a)+d(x,a,z)+d(a,y,z),$ for any $x,y,z,a\in M$.

2010 *Mathematics Subject Classification.* 54A10, 54E35, 54E99
Key words and phrases. $(3,1,\rho)-$metric, $(3,1,\rho)-$metrizable spaces, continuity of $(3,1,\rho)-$metric *d*

For a map d as above let $\rho = \{(x,y,z) \mid (x,y,z) \in M^3, d(x,y,z) = 0\}$. The set ρ is a $(3,1)$-equivalence on M, as defined and discussed in [6], [4]. The sets $\Delta = \{(x,x,x) \mid x \in M\}$ and $\nabla = \{(x,x,y) \mid x,y \in M\}$ are (3,1)-equivalences on M. The condition $(\boldsymbol{M}0)$ implies that $\Delta \subseteq \rho$.

Definition 1. Let $d: M^3 \to R_0^+$ and ρ be as above. If d satisfies $(\boldsymbol{M}0)$, $(\boldsymbol{P})$ and $(\boldsymbol{M}1)$ we say that d is a $(3,1,\rho)-$*metric* on M.

Let d be a $(3,1,\rho)-$metric on M, $x,y \in M$ and $\varepsilon > 0$. As in [4], we consider the following $\varepsilon-$ball, as subset of M:

$B(x,y,\varepsilon) = \{z \mid z \in M, d(x,y,z) < \varepsilon\}$ - ε-ball with center at (x,y) and radius ε.

Among the others, a $(3,1,\rho)-$metric d on M induces the topology $\tau(G,d)$ - generated by all the ε-balls $B(x,y,\varepsilon)$, i.e. the topology whose base is the set of the finite intersections of ε-balls $B(x,y,\varepsilon)$, (see as [4]) .

2. CONTINUITY OF A $(3,1,\rho)-$METRIC d FOR $\tau(G,d)$

Proposition 1. *Let d be a $(3,1,\rho)-$metric on M,* let $\tau = \tau(G,d)$, *and let d satisfies the following two conditions:*

(A) *For each x_1, x_2, x_3 of M there is permutation i_1, i_2, i_3 of* 1,2,3 *such that*

$$d(x_{i_1}, x_{i_1}, x_{i_2}) = d(x_{i_1}, x_{i_1}, x_{i_3}) = d(x_{i_2}, x_{i_3}, x_{i_3}) = 0 \text{ and}$$

(B) *For each two points u,v of M and each $\varepsilon > 0$ there are open sets $U_u, U_v \in \tau$ such that $u \in U_u$, $v \in U_v$ and for each $x \in U_u$ and $y \in U_v$:*

$$d(u,x,y) < \varepsilon \text{ and } d(v,x,y) < \varepsilon.$$

Then, the $(3,1,\rho)-$metric d is a continuous function.

Proof. Let u,v,t be points of M and $\varepsilon > 0$. Using (A), w.l.o.g. we can set $x_1 = u$, $x_2 = t$ and $x_3 = v$. Thus:

$$d(u,u,t) = d(u,u,v) = d(t,v,v) = 0. \tag{1}$$

For u,v of M and $\varepsilon > 0$, the condition (B) implies that there are open neighborhoods U_u, U_v of *u* and *v*, such that for each x of U_u and each y of U_v we have

$$d(x,y,u) < \varepsilon/6 \text{ and } d(x,y,v) < \varepsilon/6. \tag{2}$$

Let U_u^1 and U_v^1 be the open sets defined by:

$$U_u^1 = B(u,v,\varepsilon/6) \cap B(u,t,\varepsilon/6) \cap U_u \text{ and } U_v^1 = B(t,v,\varepsilon/6) \cap U_v. \tag{3}$$

Then, (1) implies that $u \in U_u^1$ and $v \in U_v^1$. This together with (3), implies that for each x of U_u^1 and each y of U_v^1, we have

$$d(u,v,x) < \varepsilon/6 \, d(u,t,x) < \varepsilon/6 \text{ and } d(t,v,y) < \varepsilon/6. \tag{4}$$

For u,t of M and $\varepsilon > 0$, the condition (B) implies that there are open neighborhoods U_u^2 of u and U_t of t such that for each x of U_u^2 and each z of U_t we have

$$d(u,x,z) < \varepsilon/6 \text{ and } d(t,x,z) < \varepsilon/6. \tag{5}$$

For t,v of M and $\varepsilon > 0$, the condition (B) implies that there are open neighborhoods U_t^1 of t and U_v^2 of v such that for each z of U_t^1 and each y of U_v^2 we have

$$d(t,z,y) < \varepsilon/6 \text{ and } d(v,z,y) < \varepsilon/6. \tag{6}$$

Let $U_u^{'} = U_u \cap U_u^1 \cap U_u^2$, $U_v^{'} = U_v \cap U_v^1 \cap U_v^2$ and $U_t^{'} = U_t \cap U_t^1$. The construction of these open sets implies that $u \in U_u^{'}$, $v \in U_v^{'}$ and $t \in U_t^{'}$. Moreover, for each x of $U_u^{'}$, y of $U_v^{'}$ and z of $U_t^{'}$, using (2), (4), (5) and (6), and the tetrahedral inequality (M1) several times we obtain the following inequalities:

$$\begin{aligned} d(u,t,v) &\le d(u,t,x) + d(u,x,v) + d(x,t,v) < \varepsilon/6 + \varepsilon/6 + d(x,t,v) \\ &\le \varepsilon/3 + d(x,t,y) + d(x,y,v) + d(y,t,v) < \varepsilon/3 + d\left(x,t,y\right) + \varepsilon/6 + \varepsilon/6 \\ &\le 2\varepsilon/3 + d(x,t,z) + d(x,z,y) + d(z,t,y) < 2\varepsilon/3 + \varepsilon/6 + d(x,z,y) + \varepsilon/6 \\ &= d(x,z,y) + \varepsilon, \end{aligned} \tag{7}$$

$$\begin{aligned} d(x,z,y) &\le d(x,z,t) + d(x,t,y) + d(t,z,y) < \varepsilon/6 + \varepsilon/6 + d(x,t,y) \\ &\le \varepsilon/3 + d(x,t,v) + d(x,v,y) + d(v,t,y) < \varepsilon/3 + d(x,t,v) + \varepsilon/6 + \varepsilon/6 \\ &\le 2\varepsilon/3 + d(x,t,u) + d(x,u,v) + d(u,t,v) < 2\varepsilon/3 + \varepsilon/6 + d(u,t,v) + \varepsilon/6 \\ &= d(u,t,v) + \varepsilon. \end{aligned} \tag{8}$$

Next, (7) and (8) imply that:

$$| d(u,t,v) - d(x,z,y) | < \varepsilon.$$

All this shows that d is a continuous function from M^3 to R with the usual topology. ■

With the next example we show the existence of a continuous $(3,1,\rho)-$metric d satisfying the condition (A), but not satisfying the condition (B) as in Proposition 1.

Example 1. Let $M = (p_1) \cup (p_2)$ whereas (p_1) and (p_2) are parallel lines and let $d : M^3 \to R_0^+$ be defined by:

$$d\left(x,y,z\right) = \begin{cases} 0, & x = y = z \text{ or } x \ne y \ne z \ne x \text{ and } x,y,z \in (p_k), k = 1,2 \\ 1, & \text{in other cases} \end{cases}$$

$$d(x,x,y) = \begin{cases} 0, & x,y \in (p_k), k = 1,2 \text{ or } x \in (p_1), y \in (p_2) \\ 1, & x \in (p_2), y \in (p_1). \end{cases}$$

It is easy to show that d is a $(3,1,\rho)-$metric on M with

$$\rho=\Delta\cup\{(x,x,y)|x\in(p_1),y\in(p_2)\text{or }x,y\in(p_k),k=1,2\}$$
$$\cup\{(x,y,z)|x\neq y\neq z\neq x\text{ and }x,y,z\in(p_k),k=1,2\}.$$

For $x\neq y\in M$ and $\varepsilon>0$,

$$B(x,y,\varepsilon)=\begin{cases}(p_k), & x\neq y,\quad x,y\in(p_k),k=1,2,\varepsilon\leq 1\\ \{x\}, & x\neq y,\quad x\in(p_1),y\in(p_2),\varepsilon\leq 1\\ M, & \varepsilon>1,\end{cases},$$

and for $x=y\in M$ and $\varepsilon>0$

$$B(x,x,\varepsilon)=\begin{cases}(p_2), & x\in(p_2),\varepsilon\leq 1\\ M, & x\in(p_1),\varepsilon>1.\end{cases}$$

From this it follows that $\tau=\tau(G,d)=D_{(p_1)}\cup\{(p_2)\cup V|V\subseteq(p_1)\}$ where $D_{(p_1)}$ is the discrete topology on (p_1).

First we show that the $(3,1,\rho)-$metric d satisfies the condition (A).

- If $x,y,z\in(p_k),k=1,2,$ then $d(x,x,y)=d(x,x,z)=d(y,z,z)=0.$
- If $x,y\in(p_1),\ z\in(p_2),$ then $d(y,y,x)=d(y,y,z)=d(x,z,z)=0.$
- If $x\in(p_1),\ \ y,z\in(p_2),$ then $d(x,x,y)=d(x,x,z)=d(y,z,z)=0.$
- If $x=z\in(p_2),\ y\in(p_1),$ then $d(x,x,x)=d(x,y,y)=0$ and $d(y,y,y)=d(y,y,x)=0.$

We will show that the $(3,1,\rho)-$metric d is a continuous function. For each x,y of M we define the map $f_{x,y}:M\to R$ by $f_{x,y}(z)=d(x,y,z).$

Let U be an open set in R with the usual topology, such that $1\in U$ and $0\notin U.$ Then

$$f^{-1}_{x,y}(U)=\{z|\ f_{x,y}(z)\in U\}=\{z|\ d(x,y,z)=1\}.$$

We consider the following cases:

- if $x\neq y$ and $x,y\in(p_1),$ then $f^{-1}_{x,y}(U)=(p_2)\in\tau,$
- if $x\neq y$ and $x,y\in(p_2),$ then $f^{-1}_{x,y}(U)=(p_1)\in\tau,$
- if $x\neq y$ and $x\in(p_1)$, $y\in(p_2),$ then $f^{-1}_{x,y}(U)=(p_1)\setminus\{x\}\cup(p_2)\in\tau,$
- if $x=y$ and $x\in(p_1)$, then $f^{-1}_{x,x}(U)=\varnothing\in\tau,$
- if $x=y$ and $x\in(p_2)$, then $f^{-1}_{x,x}(U)=(p_1)\in\tau.$

Let V be an open set in R such that $0\in V, 1\notin V$. Then

$$f^{-1}_{x,y}(V)=\{z|\ f_{x,y}(z)\in V\}=\{z|\ d(x,y,z)=0\}.$$

We consider the following cases:

- if if $x\neq y$ and $x,y\in(p_1),$ then $f^{-1}_{x,y}(V)=(p_1)\in\tau,$

- if $x \neq y$ and $x, y \in (p_2)$, then $f_{x,y}^{-1}(V) = (p_2) \in \tau$,

- if $x \neq y$ and $x \in (p_1)$, $y \in (p_2)$, then $f_{x,y}^{-1}(V) = \{x\} \in \tau$,

- if $x = y$ and $x \in (p_1)$, then $f_{x,x}^{-1}(V) = M \in \tau$,

- if $x = y$ and $x \in (p_2)$, then $f_{x,x}^{-1}(V) = (p_2) \in \tau$.

Let W be an open set in R such that $0,1 \in W$. Then

$$f_{x,y}^{-1}(W) = \{z | f_{x,y}(z) \in W\} = \{z \,|\, d(x,y,z) = 0 \text{ or } d(x,y,z) = 1\} = M \in \tau.$$

All this implies that *d* is a continuous function.

Next, we show that d does not satisfy the condition (B) from the above proposition. Let $u, v \in M$, and let $u \in (p_1)$, $v \in (p_2)$ and $0 < \varepsilon < 1$. For each open neighborhoods U_u of u and U_v of v, for $x = u$ and each $y = v$ we have

$$d(u,x,y) = d(u,u,v) = 0 < \varepsilon \text{ and } d(v,x,y) = d(u,v,v) = 1 \not< \varepsilon.$$

Hence, the condition (B) is not satisfied.

References

[1] П.С.Александров, В.В.Немыцкий, *Условия метризуемости топологических пространств и аксиома симетрии,* Мат. сб. 3:3 (1938), 663-672

[2] А.В. Архангельский, *О поведении метризуемости при факторных тображениях,* ДАН 164, No 2 (1965), 247-250

[3] М. Чобан, *О метрисуемыих пространствах,* Вестн.Моск. Ун-та, сер.Матем., мех, No 3 (1959), 44-50

[4] S. Čalamani, D. Dimovski, *Topologies induced by (3,1,ρ)-metrics and (3,2,ρ)-metrics,* Inernational mathematical forum, Vol.9, no.21-24, (2014) 1075-1088

[5] B.C. Dhage, *Generalized metric spaces and topological structure II,* Pure Appl.Math. Sci., 40 (1-2) (1994), 37-41

[6] D. Dimovski, *Generalized metrics - (n,m,r)-metrics,* Mat. Bilten, 16, Skopje (1992), 73-76

[7] D. Dimovski, *(3,1,ρ)-metrizable topological spaces*, Math. Macedonica, 3 (2005), 59-64

[8] S. Gähler, *2-metrische räume und ihre topologische Struktur,* Math. Nachr. 26 (1963), 115-148

[9] R. Kopperman, *All topologies come from generalized metrics,* A. Math. Month. V 95 i2, 89-97

[10] Z. Mamuziċ, *Abstract distance and neighborhood spaces,* Proc. Prague Symp. (1962), 261-266

[11] K. Menger, *Untersuchungen über allgemeine Metrik,* Math. Ann. 100 (1928), 75-163

[12] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, Journal of Nonlinear and Convex Analysis, Vol. 7, Number 2 (2006), 289-297

[13] V. Nemytzki, *On the "third axiom of metric spaces"*, Tr. Amer. Math. Soc. 29 (1927), 507-513

[14] С. Недев, *о-Метризуемые пространства*, Тр. Моск. Мат. Общ. Том 24 (1971), 201-236

[15] J. Usan, *<Nm,E>-seti s (n+1)-rastojaniem*, Review of Research, PMF, Novi Sad, Ser. Mat. 17 (1989), 2, 65-87

[1)] Faculty of technical sciences, Univ. "St. Kliment Ohridski", Bitola, Macedonia
E-mail address: scalamani@yahoo.com
[2)] Faculty of Natural Sciences & Mathematics, Ss. Cyril and Methodius, Skopje, Macedonia
E-mail address: donco@pmf.ukim.mk