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WEIGHTED NORLUND-EULER A-STATISTICAL CONVERGENCE FOR SEQUENCES OF POSITIVE LINEAR OPERATORS

Elida Hoxha[1], Ekrem Aljimi[2] and Valdete Loku[3]

Abstract. We introduce the notion of weighted Norlund –Euler A-Statistical Convergence of a sequence, where A represents the nonnegative regular matrix. We also prove the Korovkin approximation theorem by using the notion of weighted Norlund-Euler A-statistical convergence. Further, we give a rate of weighted Norlund-Euler A-statistical convergence.

1. BACKGROUND, NOTATIONS AND PRELIMINARIES

Suppose that $E \subseteq N = \{1,2,\ldots\}$ and $E_n = \{k \le n : k \in E\}$. Then

$$\delta(E) = \lim_{n\to\infty} \tfrac{1}{n}|E_n| \tag{1}$$

is called the natural density of E provided that the limit exist, where $|.|$ represents the number of elements in the enclosed set.

The term "statistical convergence" was first presented by Fast [1] which is generalization of the concept of ordinary convergence. Actually, a root of the notion of statistical convergence can be detected by Zygmund [2] (also see [3]), where he used the term 'almost convergence' which turned out to be equivalent to the concept of statistical convergence. The notion of Fast was further investigated by Schoenberg [4], Salat [5], Fridy [6], and Conner [7].

The following notion is due to Fast [1]. A sequence $x = (x_k)$ is said to be statistically convergent to L if $\delta(K_\varepsilon) = 0$ for every $\varepsilon > 0$, where

$$K_\varepsilon = \{k \in N : |x_k - L| \ge \varepsilon\} \tag{2}$$

equivalently,

$$\lim_{n\to\infty} \tfrac{1}{n}\left|\{k \le n : |x_k - L| \ge \varepsilon\}\right| = 0\,. \tag{3}$$

In symbol, we will write $S - \lim x = L$. We remark that every convergent sequence is statistically convergent but not conversely.

Let X and Y be two sequence spaces and let $A = (a_{n,k})$ be an infinite matrix. If for each $x = (x_k)$ in X the series

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$$A_n x = \sum_k a_{n,k} x_k = \sum_{k=1}^{\infty} a_{n,k} x_k \tag{4}$$

converges for each $n \in N$ and the sequence $Ax = A_n x$ belongs to Y, then we say the matrix A maps X to Y. By the symbol (X,Y) we denote the set of all matrices which map X into Y.

A matrix A (or a matrix map A) is called regular if $A \in (c,c)$, where the symbol c denotes the spaces of all convergent sequences and

$$\lim_{n\to\infty} A_n x = \lim_{k\to\infty} x_k \tag{5}$$

for all $x \in c$. The well-known Silverman-Toeplitz theorem (see [8]) assert that $A = (a_{n,k})$ is regular if and only if

i) $\lim_n a_{n,k} = 0$ for each k;

ii) $\lim_n \sum_k a_{n,k} = 1$;

iii) $\sup_n \sum_k |a_{n,k}| < \infty$.

Kolk [9] extended the definition of statistical convergence which the help of nonnegative regular matrix $A = (a_{n,k})$ calling it A-statistical convergence. The definition of A-statistical convergence is given by Kolk as follows. For any nonnegative regular matrix A, we say that a sequence is A-statistically convergent to L provided that for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \sum_{k:|x_k - L| \geq \varepsilon} a_{n,k} = L \tag{6}$$

In 2009, the concept of weighted statistical convergence was defined and studied by Karakaya and Chishti [10] and further modified by Mursaleen et al. [11] in 2012. In 2013, Belen and Mohiuddine [12] presented a generalization of this notion through de la Vallee-Poussin mean in probabilistic normed spaces.

Let $\sum_{k=0}^{n} x_n$ be a given infinite series with sequence of its n^{th} partial sum $\{S_n\}$. If $(E,1)$ transform is defined as

$$E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} S_k \tag{7}$$

and we say that this summability method is convergent if $E_n^1 \to S$ as $n \to \infty$. In this case we say the series $\sum_{k=0}^{n} x_n$ is $(E,1)$ – summable to a definite number S. (Hardy [31]). And we will write $S_n \to S\left(E,1\right)$ as $n \to \infty$.

Let (p_n) and (q_n) be the two sequences of non-zero real constants such that

$$P_n = p_0 + p_1 + \ldots + p_n,\ P_{-1} = p_{-1} = 0$$
$$Q_n = q_0 + q_1 + \ldots + q_n,\ Q_{-1} = q_{-1} = 0$$

For the given sequences (p_n) and (q_n), convolution $p*q$ is defined by:

$$R_n = p*q = \sum_{k=0}^{n} p_n q_{n-k}\,. \tag{8}$$

The series $\sum_{k=0}^{n} x_n$ or the sequence $\{S_n\}$ is summable to S by generalized Norlund method and it is denoted by $S_n \to S(N,p,q)$ if

$$t_n^{p,q} = \frac{1}{R_n}\sum_{v=0}^{n} p_{n-v} q_v S_v \tag{9}$$

tends to S as $n \to \infty$.

Let us use in consideration the following method of summability:

$$t_n^{p,q,E} = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k E_k^1 = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k \frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v} S_v \tag{10}$$

If $t_n^{p,q,E} \to S$ as $n \to \infty$, then we say that the series $\sum_{k=0}^{n} x_n$ or the sequence $\{S_n\}$ is summable to S by Norlund-Euler method and it is denoted by $S_n \to S\left(N,p,q\right)(E,1)$.

Remark 1. If $p_k = 1, q_k = 1$, then we get Euler summability method.

Now we are able to give the definition of the weighted statistical convergence related to the $(N,p,q)(E,1)$ – summability method.

We say that E have weighted density, denoted by $\delta_{NE}(E)$, if

$$\delta_{NE}\left(E\right) = \lim_{n\to\infty}\frac{1}{R_n}\left|\{k \le R_n : k \in E\}\right|. \tag{11}$$

A sequence $x = (x_k)$ is said to be weighted Norlund-Euler statistical convergent (or S_{NE} – convergent) if for every $\varepsilon > 0$:

$$\lim_{n\to\infty}\frac{1}{R_n}\left|\{k \le R_n : p_{n-k} q_k \frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}\left|x_v - L\right| \ge \varepsilon\}\right| = 0 \tag{12}$$

In these case we write $L = S_{NE}\left(st\right) - \lim x$.

In the other hand, let us recall that $C[a,b]$ is the space of all functions f continuous on $[a,b]$. We know that $f \in C[a,b]$ is Banach spaces with norm

$$\|f\|_\infty = \sup_{x\in[a,b]}|f(x)|,\quad f \in C[a,b] \tag{13}$$

Suppose that L is a linear operator from $C[a,b]$ into $C[a,b]$. It is clear that if $f \geq 0$ implies $Lf \geq 0$, then the linear operator L is positive on $C[a,b]$. We denote the value of Lf at a point $x \in [a,b]$ by $L(f;x)$. The classical Korovkin approximation theorem states the following [14].

Theorem 2. Let (T_n) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then,

$$\lim_{n\to\infty} \| T_n(f;x) - f(x) \|_\infty = 0 \tag{14}$$

for all $C[a,b]$ if only if

$$\lim_{n\to\infty} \| T_n\left(f_i;x\right) - f_i(x) \|_\infty = 0 \tag{15}$$

where $f_i\left(x\right) = x^i$ and $i = 0,1,2.$

Many mathematicians extended the Korovkin-type approximation theorems by using various test functions in several setups, including Banach spaces, abstract Banach lattices, function spaces, and Banach algebras. Firstly, Gadjiev and Orhan [15] established classical Korovkin theorem through statistical convergence and display an interesting example in support of our result. Recently, Korovkin-type theorems have been obtained by Mohiuddine [16] for almost convergence. Korovkin-type theorems were also obtained in [17] for λ-statistical convergence. The authors of [18] established these types of approximation theorem in weighted L_p spaces, where $1 \leq p < \infty$, through A-summability which is stronger than ordinary convergence. For these types of approximation theorems and related concepts, one can be referred to [19–29] and references therein.

2. KOROVKIN-TYPE THEOREMS BY WEIGHTED NORLUND-EULER A-STATISTICAL CONVERGENCE

Kolk [9] introduced the notion of A -statistical convergence by considering nonnegative regular matrix A instead of Cesáro matrix in the definition of statistical convergence due to Fast. Inspired from the work of S. A. Mohiuddine, Abdullah Alotaibi, and Bipan Hazarika [30] we introduce the notion of weighted Norlund-Euler A-statistical convergence of a sequence and then we establish some Korovkin-type theorems by using this notion.

Definition 3. Let $A = (a_{n,k})$ be a nonnegative regular matrix. A sequence $x = (x_k)$ of real or complex numbers is said to be weighted Norlund –Euler A-statistical convergence, denoted by S_A^{NE} –convergent, to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \sum_{k\in E(p,\varepsilon)} a_{n,k} = 0 \tag{16}$$

where

$$E(p,\varepsilon) = \{k \in N : p_{n-k} q_k \tfrac{1}{2^k} \sum_{v=0}^{k} \tbinom{k}{v} | x_v - L | \geq \varepsilon\} \tag{17}$$

In symbol, we will write $S_A^{NE} - \lim x = L$.

Remark 4. Note that convergence sequence implies weighted Norlund-Euler A-statistical convergent to the same value but converse is not true in general. For example, take $p_k = 1, q_k = 1$ for all k and define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} 1, & \text{if } k = n^2 \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

where $n \in N$. Then this sequence is statistically convergent to 0 but not convergent; in this case, weighted Norlund-Euler A-statistical convergence of a sequence coincides with statistical convergence.

Theorem 5. Let $A = (a_{n,k})$ be a nonnegative regular matrix. Consider a sequence of positive linear operators (M_k) from $C[a,b]$ into itself. Then, for all $f \in C[a,b]$ bounded on whole real line,

$$S_A^{NE} - \lim_{k\to\infty} \| M_k\left(f; x\right) - f(x) \|_\infty = 0 \tag{19}$$

if only if

$$\begin{aligned} &S_A^{NE} - \lim_{k\to\infty} \| M_k\left(1; x\right) - 1 \|_\infty = 0, \\ &S_A^{NE} - \lim_{k\to\infty} M_k \|\left(v; x\right) - x \|_\infty = 0, \\ &S_A^{NE} - \lim_{k\to\infty} \| M_k(v^2; x) - x^2 \|_\infty = 0 \end{aligned} \tag{20}$$

Proof. Equation (20) directly follows from (19) because each of $1, x, x^2$ belongs to $C[a,b]$. Consider a function $f \in C[a,b]$. Then there is a constant $C > 0$ such that $|f(x)| \leq C$ for all $x \in (-\infty, +\infty)$. Therefore,

$$\left| f\left(v\right) - f(x) \right| \leq 2C \ , \quad -\infty < v, x < +\infty \, , \tag{21}$$

Let $\varepsilon > 0$ be given. By hypothesis there is a $\delta = \delta(\varepsilon) > 0$ such that

$$| f\left(v\right) - f(x) | < \varepsilon, \qquad \forall \, | v - x | < \delta \tag{22}$$

Solving (21) and (22) and then substituting $\Omega(v) = (v - x)^2$, one obtains

$$| f\left(v\right) - f(x) | < \varepsilon + \tfrac{2C}{\delta^2} \Omega \, . \tag{23}$$

Equation (23) can be also written by as

$$-\varepsilon-\frac{2C}{\delta^2}\Omega< f(\nu)-f(x)<\varepsilon+\frac{2C}{\delta^2}\Omega\,. \tag{24}$$

Operating $M_k(1;x)$ to (24) since $M_k(f;x)$ his linear and monoton, one obtains

$$M_k(1;x)(-\varepsilon-\frac{2C}{\delta^2}\Omega)< M_k(1;x)(f(v)-f(x))< M_k(1;x)(\varepsilon+\frac{2C}{\delta^2}\Omega) \tag{25}$$

Note that x is fixed, so $f(x)$ is constant number. Thus, we obtain from (25) that

$$-\varepsilon M_k(1;x)-\frac{2C}{\delta^2}M_k(\Omega;x)< M_k(f;x)-f(x)M_k(1;x)<\varepsilon M_k(1;x)+\frac{2C}{\delta^2}M_k(\Omega;x) \tag{26}$$

The term "$M_k(f;x)-f(x)M_k(1;x)$" in (26) can also written as

$$M_k(f;x)-f(x)M_k(1;x)=M_k(f;x)-f(x)-f(x)[M_k(1;x)-1] \tag{27}$$

Now substituting the value of $M_k(f;x)-f(x)M_k(1;x)$ in (26), we get that

$$M_k(f;x)-f(x)<\varepsilon M_k(1;x)+\frac{2C}{\delta^2}M_k(\Omega;x)+f(x)[M_k(1;x)-1] \tag{28}$$

We can rewrite the term "$M_k(\Omega;x)$" in (28) as follows:

$$\begin{aligned} M_k(\Omega;x)&=M_k((\nu-x)^2;x)=M_k(\nu^2;x)+2xM_k(\nu;x)+x^2M_k(1;x)\\ &=[M_k(\nu^2;x)-x^2]-2x\big[M_k(\nu;x)-x\big]+x^2[M_k(1;x)-1] \end{aligned} \tag{29}$$

Equation (28) with the above value of $M_k(\Omega;x)$ becomes

$$\begin{aligned} M_k(f;x)-f(x)&<\varepsilon M_k(1;x)+\frac{2C}{\delta^2}\{[M_k(\nu^2;x)-x^2]+2x[M_k(\nu;x)-x]\\ &\quad+x^2[M_k(1;x)-1]\}+f(x)[M_k(1;x)-1]\\ &=\varepsilon[M_k(1;x)-1]+\varepsilon+\frac{2C}{\delta^2}\{[M_k(\nu^2;x)-x^2]+2x[M_k(\nu;x)-x]\\ &\quad+x^2[M_k(1;x)-1]\}+f(x)[M_k(1;x)-1] \end{aligned} \tag{30}$$

Therefore,

$$\begin{aligned} \left|M_k(f;x)-f(x)\right|&\le(\varepsilon+\frac{2Cb^2}{\delta^2}+C)\left|M_k(1;x)-1\right|\\ &\quad+\frac{2C}{\delta^2}\,|\,M_k(\nu^2;x)-x^2\,|\\ &\quad+\frac{4Cb}{\delta^2}\,|\,M_k(\nu;x)-x\,| \end{aligned} \tag{31}$$

where $b=\max|x|$. Taking supremum over $x\in[a,b]$, one obtains

$$\begin{aligned} \|M_k(f;x)-f(x)\|_\infty&\le(\varepsilon+\frac{2Cb^2}{\delta^2}+C)\|M_k(1;x)-1\|_\infty\\ &\quad+\frac{2C}{\delta^2}\|M_k(\nu^2;x)-x^2\|_\infty\\ &\quad+\frac{4Cb}{\delta^2}\|M_k(\nu;x)-x\|_\infty \end{aligned} \tag{32}$$

or

$$\begin{aligned} \|M_k(f;x)-f(x)\|_\infty&\le T\{\|M_k(1;x)-1\|_\infty\\ &\quad+\|M_k(\nu^2;x)-x^2\|_\infty\\ &\quad+\|M_k(\nu;x)-x\|_\infty\} \end{aligned} \tag{33}$$

where

$$T = \max\{\varepsilon + \frac{2Cb^2}{\delta^2} + C, \frac{2C}{\delta^2}, \frac{4Cb}{\delta^2}\}. \tag{34}$$

Hence

$$\begin{aligned} p_{n-k} q_k \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \| M_k(f;x) - f(x) \|_\infty \leq T\{ p_{n-k} q_k \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \| M_k(1;x) - 1 \|_\infty \\ + p_{n-k} q_k \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \| M_k(\nu^2;x) - x^2 \|_\infty \\ + p_{n-k} q_k \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \| M_k(\nu;x) - x \|_\infty \} \end{aligned} \tag{35}$$

For given $\alpha > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \alpha$, and we will define the following sets:

$$\begin{aligned} E &= \{k \in N : p_{n-k} q_k \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \| M_k(f;x) - f(x) \|_\infty \geq \alpha\} \\ E_1 &= \{k \in N : p_{n-k} q_k \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \| M_k(1,x) - 1 \|_\infty \geq \frac{\alpha-\varepsilon}{3T}\} \\ E_2 &= \{k \in N : p_{n-k} q_k \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \| M_k(\nu;x) - x \|_\infty \geq \frac{\alpha-\varepsilon}{3T}\} \\ E_3 &= \{k \in N : p_{n-k} q_k \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \| M_k(\nu^2;x) - x^2 \|_\infty \geq \frac{\alpha-\varepsilon}{3T}\} \end{aligned} \tag{36}$$

It easy to see that

$$E \subset E_1 \cup E_2 \cup E_3 \tag{37}$$

Thus, for each $n \in N$, we obtain from (35) that

$$\sum_{k \in E} a_{n,k} \leq \sum_{k \in E_1} a_{n,k} + \sum_{k \in E_2} a_{n,k} + \sum_{k \in E_3} a_{n,k} \tag{38}$$

Taking limit $n \to \infty$ in (38) and also (20) gives that

$$\lim_{n \to \infty} \sum_{k \in E} a_{n,k} = 0. \tag{39}$$

These yields that

$$S_A^{NE} - \lim_{k \to \infty} \| M_k(f;x) - f(x) \|_\infty = 0 \tag{40}$$

for all $f \in C[a,b]$.

We also obtain the following Korovkin-type theorem for weighted Norlund-Euler statistical convergence instead of nonnegative regular matrix A in Theorem 5.

Theorem 6. Consider a sequence of positive linear operators (M_k) from $C[a,b]$ into itself. Then, for all $f \in C[a,b]$ bounded on whole real line,

$$S_{NE} - \lim_{k \to \infty} \| M_k(f;x) - f(x) \|_\infty = 0 \tag{41}$$

if only if

$$S_{NE}-\lim_{k\to\infty}\|M_k(1;x)-1\|_\infty=0 \tag{42}$$

$$S_{NE}-\lim_{k\to\infty}\|M_k(v;x)-x\|_\infty=0 \tag{43}$$

$$S_{NE}-\lim_{k\to\infty}\|M_k(v^2;x)-x^2\|_\infty=0 \tag{44}$$

Proof. Following the proof of Theorem 5, one obtains

$$E\subset E_1\cup E_2\cup E_3 \tag{45}$$

and so

$$\delta_{NE}(E)\subset\delta_{NE}(E_1)+\delta_{NE}(E_2)+\delta_{NE}(E_3) \tag{46}$$

Equations (42)-(44) give that

$$S_{NE}-\lim_{k\to\infty}\|M_k(f;x)-f(x)\|_\infty=0\,. \tag{47}$$

Remark 7. By the Theorem 2 of [32], we have that if a sequence $x=(x_k)$ is weighted Norlun-Euler statistically convergent to L, then it is strongly $(N,p,q)(E,1)-$ summable to L, provided that $p_{n-k}q_k\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}|x_k-L|$ is bounded; that is, there exist a constant C such that

$$p_{n-k}q_k\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}|x_k-L|\le C$$

for all $k\in N$. We write

$$\left|(N,p,q)(E,1)\right|=\{x=x_n:\lim_{n\to\infty}\frac{1}{R_n}\sum_{k=0}^{n}p_{n-k}q_k\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}|x_v-L|=0\text{ for some }L\} \tag{48}$$

the set of all sequences $x=(x_k)$ which are strongly $(N,p,q)(E,1)-$ summable to L.

Theorem 8. Let $M_k:C[a,b]\to C[a,b]$ be a sequence of positive linear operators which satisfies (43)-(44) of Theorem 6 and the following conditions holds:

$$\lim_{k\to\infty}\|M_k(1;x)-1\|_\infty=0\,. \tag{49}$$

Then,

$$\lim_{n\to\infty}\frac{1}{R_n}\sum_{k=0}^{n}p_{n-k}q_k\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}\|M_k(f;x)-f(x)\|_\infty=0, \tag{50}$$

for any $f\in C[a,b]$.

Proof. It follow from (49) that $\|M_k(f;x)\|_\infty\le C'$, for some constant $C'>0$ and for all $k\in N$. Hence for $f\in C[a,b]$, one obtains

$$p_{n-k}q_k \frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\| M_k(f;x)-f(x)\|_\infty \le p_{n-k}q_k \frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}(\| f\|_\infty \| M_k(1;x)\|_\infty + \| f\|_\infty)$$
$$\le p_{n-k}q_k \frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}C(C'+1). \tag{51}$$

Right hand side of (51) is constant, so

$$p_{n-k}q_k \frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\| M_k(f;x)-f(x)\|_\infty$$

is bounded. Since (49) implies (42), by Theorem 6 we get that

$$S_{NE}-\lim_{k\to\infty}\| M_k(f;x)-f(x)\|_\infty = 0\,. \tag{52}$$

By remark 7, (51) and (52) together give the desired result.

3. RATE OF WEIGHTED NORLUND-EULER A-STATISTICAL CONVERGENCE

First we define the rate of weighted Norlund-Euler A-statistical convergent sequence as follows.

Definition 9. Let $A=(a_{n,k})$ be a nonnegative regular matrix and let (a_k) be a positive non increasing sequence. Then, a sequence $x=(x_k)$ is weighted Norlund-Euler A-statistical convergent to L with the rate of $o(a_k)$ if for each $\varepsilon>0$

$$\lim_{n\to\infty}\frac{1}{a_n}\sum_{k\in E(p,\varepsilon)} a_{n,k}=0 \tag{53}$$

where

$$E(p,\varepsilon)=\{k\in N: p_{n-k}q_k \frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}|x_\nu - L|\ge \varepsilon\} \tag{54}$$

In symbol, we will write

$$x_k - L = S_A^{NE} - o(a_k) \text{ as } k\to\infty \tag{55}$$

We will prove the following auxiliary result by using the above definition.

Lemma 10. Let $A=(a_{n,k})$ be a nonnegative regular matrix. Suppose that (a_k) and (b_k) are two positive nonincreasing sequences. Let $x=(x_k)$ and $y=(y_k)$ be two sequences such that

$$x_k - L_1 = S_A^{NE} - o(a_k) \text{ and } y_k - L_2 = S_A^{NE} - o(b_k)\,.$$

Then,

(i) $(x_k - L_1)\pm(y_k - L_2) = S_A^{NE} - o(c_k)$,

(ii) $(x_k - L_1)(y_k - L_2) = S_A^{NE} - o(a_k b_k)$,

(iii) $\alpha(x_k - L_1) = S_A^{NE} - o(a_k)$, for any scalar α,

where $c_k = \max\{a_k, b_k\}$.

Proof. Suppose that

$$x_k - L_1 = S_A^{NE} - o(a_k), \quad y_k - L_2 = S_A^{NE} - o(b_k) \tag{56}$$

Given $\varepsilon > 0$, define

$$E' = \{k \in N : p_{n-k} q_k \tfrac{1}{2^k} \sum_{\nu=0}^{k} \tbinom{k}{\nu} |(x_k - L_1) \pm (y_k - L_2)| \geq \varepsilon\}$$

$$E'' = \{k \in N : p_{n-k} q_k \tfrac{1}{2^k} \sum_{\nu=0}^{k} \tbinom{k}{\nu} |x_k - L_1| \geq \tfrac{\varepsilon}{2}\} \tag{57}$$

$$E''' = \{k \in N : p_{n-k} q_k \tfrac{1}{2^k} \sum_{\nu=0}^{k} \tbinom{k}{\nu} |y_k - L_2| \geq \tfrac{\varepsilon}{2}\}$$

It easy to see that

$$E' \subset E'' \cup E''' \tag{58}$$

These yields that

$$\frac{1}{c_n} \sum_{k \in E'} a_{n,k} \leq \frac{1}{c_n} \sum_{k \in E''} a_{n,k} + \frac{1}{c_n} \sum_{k \in E'''} a_{n,k} \tag{59}$$

holds for all $n \in N$. Since $c_k = \max\{a_k, b_k\}$, (59) gives that

$$\frac{1}{c_n} \sum_{k \in E'} a_{n,k} \leq \frac{1}{a_n} \sum_{k \in E''} a_{n,k} + \frac{1}{b_n} \sum_{k \in E'''} a_{n,k} \tag{60}$$

Taking limit $n \to \infty$ in (60) together with (56), we obtain

$$\lim_{n \to \infty} \frac{1}{c_n} \sum_{k \in E'} a_{n,k} = 0 \tag{61}$$

Thus,

$$(x_k - L_1) \pm (y_k - L_2) = S_A^{NE} - o(c_k) \tag{62}$$

Similarly, we can prove (ii) and (iii).

Now, we recall the notion of modulus of continuity of f in $C[a,b]$ is defined by

$$\omega(f, \delta) = \sup\{|f(x) - f(y)| : x, y \in [a,b], |x - y| < \delta\} \tag{63}$$

It is well known that

$$|f(x) - f(y)| \leq \omega(f, \delta)\left(\frac{|x-y|}{\delta} + 1\right). \tag{64}$$

Theorem 11. Let $A = (a_{n,k})$ be a nonnegative regular matrix. If the sequence of positive linear operators $M_k : C[a,b] \to C[a,b]$ satisfies the conditions

(i) $\|M_k(1;x) - 1\|_\infty = S_A^{NE} - o(a_k)$,

(ii) $\omega(f, \lambda_k) = S_A^{NE} - o(b_k)$, with $\lambda_k = \sqrt{M_k(\varphi_x; x)}$ and $\varphi_x(y) = (y - x)^2$, where (a_k) and (b_k) are two positive nonincreasing sequences, then

$$\| M_k(f;x)-f(x)\|_\infty = S_A^{NE} - o(c_k) \tag{65}$$

for all $f \in C[a,b]$, where $c_k = \max\{a_k, b_k\}$.

Proof. Equation (27) can be reformed into the following form:

$$\begin{aligned}
|M_k(f;x)-f(x)| &\le M_k(|f(x)-f(y)|;x)+|f(x)|\cdot|M_k(1;x)-1| \\
&\le M_k(1+\tfrac{|y-x|}{\delta};x)\omega(f,\delta)+|f(x)|\cdot|M_k(1;x)-1| \\
&\le M_k(1+\tfrac{(y-x)^2}{\delta^2};x)\omega(f,\delta)+|f(x)|\cdot|M_k(1;x)-1| \\
&\le (M_k(1;x)+\tfrac{1}{\delta^2}M_k(\varphi_x;x))\omega(f,\delta)+|f(x)|\cdot|M_k(1;x)-1| \\
&\le |M_k(1;x)-1|\,\omega(f,\delta)+|f(x)|\cdot|M_k(1;x)-1|+\omega(f,\delta)+\tfrac{1}{\delta^2}M_k(\varphi_x;x)\omega(f,\delta)
\end{aligned} \tag{66}$$

Choosing $\delta = \lambda_k = \sqrt{M_k(\varphi_x;x)},$ one obtains

$$\| M_k(f;x)-f(x)\|_\infty \le T\| M_k(1;x)-1\|_\infty + 2\omega(f,\lambda_k)+\| M_k(1;x)-1\|_\infty\, \omega(f,\lambda_k) \tag{67}$$

where $T = \| f\|_\infty$. For given $\varepsilon > 0,$ we will define the following sets:

$$\begin{aligned}
E_1' &= \{k \in N : p_{n-k}q_k \tfrac{1}{2^k}\sum_{\nu=0}^{k}\tbinom{k}{\nu}\| M_k(f;x)-f(x)\|_\infty \ge \varepsilon\} \\
E_2' &= \{k \in N : p_{n-k}q_k \tfrac{1}{2^k}\sum_{\nu=0}^{k}\tbinom{k}{\nu}\| M_k(1,x)-1\|_\infty \ge \tfrac{\varepsilon}{3T}\} \\
E_3' &= \{k \in N : p_{n-k}q_k \tfrac{1}{2^k}\sum_{\nu=0}^{k}\tbinom{k}{\nu}\omega(f,\lambda_k) \ge \tfrac{\varepsilon}{6}\} \\
E_4' &= \{k \in N : p_{n-k}q_k \tfrac{1}{2^k}\sum_{\nu=0}^{k}\tbinom{k}{\nu}\omega(f,\lambda_k)\| M_k(1;x)-1\|_\infty \ge \tfrac{\varepsilon}{3}\}.
\end{aligned} \tag{68}$$

It follow from (67) that

$$\frac{1}{c_n}\sum_{k\in E_1'} a_{n,k} \le \frac{1}{c_n}\sum_{k\in E_2'} a_{n,k} + \frac{1}{c_n}\sum_{k\in E_3'} a_{n,k} + \frac{1}{c_n}\sum_{k\in E_4'} a_{n,k} \tag{69}$$

holds for $n \in N$. Since $c_k = \max\{a_k, b_k\}$, we obtain from (69) that

$$\frac{1}{c_n}\sum_{k\in E_1'} a_{n,k} \le \frac{1}{a_n}\sum_{k\in E_2'} a_{n,k} + \frac{1}{b_n}\sum_{k\in E_3'} a_{n,k} + \frac{1}{c_n}\sum_{k\in E_4'} a_{n,k}\,. \tag{70}$$

Taking limit $n \to \infty$ in (70) together with Lemma 10 and our hypotheses (i) and (ii), one obtains

$$\lim_{n\to\infty}\frac{1}{c_n}\sum_{k\in E_1'} a_{n,k} = 0 \tag{71}$$

These yields

$$\| M_k(f;x)-f(x)\|_\infty = S_A^{NE} - o(c_k) \tag{72}$$

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[1] Department of Mathematics, University of Tirana, Albania.
E-mail address: hoxhaelida@yahoo.com
[2] Department of Mathematics, University of Tirana, Albania.
E-mail address:ekremhalimii@yahoo.co.uk
[3] Department of Computer Sciences and Applied Mathematics,
College, Vizioni per Arsim, Ahmet Kaciku, Nr=3, Ferizaj