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CHARACTERIZATION OF $(2k, k)$ – RECTANGULAR BAND

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Abstract

A $(2k, k)$ – semigroup $(Q; [\,])$ which is a direct product of a left-zero $(2k, k)$ – semigroup and of a right-zero $(2k, k)$ – semigroup is called $(2k, k)$– rectangular band. In this paper we give a characterization of $(2k, k)$-rectangular band.

1. Introduction

The pair $(Q; [\,])$ is called a $(2k, k)$ – semigroup if $[\,] : Q^{2k} \to Q^k$ is a map satisfying the following condition:

$$[x_1^i[x_{i+1}^{2k+i}]x_{2k+i+1}^{3k}] = [[x_1^{2k}]x_{2k+1}^{3k}], \quad \text{for each} \quad 1 \leq i \leq k,$$

where $k \geq 1$ and $Q \neq \emptyset$.

A pair $(A; [\,])$, where $[\,]$ is a $(2k, k)$ – operation defined by $[x_1^{2k}] = x_1^k$ is $(2k, k)$ – semigroup. It is called left-zero $(2k, k)$ – semigroup. Dually, a right-zero $(2k, k)$ – semigroup $(B; [\,])$ is defined by $[x_1^{2k}] = x_{k+1}^{2k}$.

A pair $(A \times B;\, [\,])$ where $[\,]$ is a $(2k, k)$ – operation on $A \times B$ defined by $[x_1^{2k}] = y_1^k \Leftrightarrow (x_i = (a_i, b_i),\, y_j = (a_j, b_{j+k}),\, i \in \mathbf{N}_{2k},\, j \in \mathbf{N}_k)$ is $(2k, k)$ – semigroup and it is a direct product of a left-zero $(2k, k)$ – semigroup and of a right-zero $(2k, k)$ – semigroup on A and B, respectively. Such a $(2k, k)$ – semigroup will be called $(2k, k)$ – rectangular band.

2. Characterization of $(2k, k)$ – rectangular band

We give a characterization of $(2k, k)$–rectangular band.

Proposition 1. Let $\mathbf{Q} = (Q; [\,])$ be a $(2k, k)$ – semigroup. $\mathbf{Q}$ is a $(2k, k)$– rectangular band if and only if the following equalities are satisfied in $\mathbf{Q}$:

2091

(1) $[abc] = [ac]$, $a, b, c \in Q^k$;
(2) $[a_1^{i-1} a a_{i+1}^k b b_{i+1}^k]_i = [x_1^{j-1} a x_{j+1}^k y_1^{j-1} b y_{j+1}^k]_j \quad i, j \in \mathbf{N}_k$;
(3) $[\overset{2k}{a}] = \overset{k}{a}$, where $\overset{i}{a}$ denotes $\underbrace{aa\ldots a}_{i}$.

Proof. Suppose a $(2k, k)$ – semigroup $\mathbf{Q} = (Q; [\,])$ satisfies (1), (2) and (3).

(A) Let a be a fixed element of Q. Denote by L the subset of Q

$$L = \{[x \overset{2k-1}{a}]_1 | x \in Q\}.$$

Let

$$[x_i \overset{2k-1}{a}]_1, [y_i \overset{2k-1}{a}]_1 \in L, i \in \mathbf{N}_k.$$

Then:

$$[[x_1 \overset{2k-1}{a}]_1 \ldots [x_k \overset{2k-1}{a}]_1 [y_1 \overset{2k-1}{a}]_1 \ldots [y_k \overset{2k-1}{a}]_1]_i \overset{(2)}{=}$$
$$= [[x_1^k \overset{k}{a}]_1 [x_1^k \overset{k}{a}]_2 \ldots [x_1^k \overset{k}{a}]_k [y_1^k \overset{k}{a}]_1 [y_1^k \overset{k}{a}]_2 \ldots [y_1^k \overset{k}{a}]_k]_i =$$
$$= [x_1^k \overset{k}{a} [y_1^k \overset{k}{a}]_1 y_1^k \overset{k}{a}]_2 \ldots [y_1^k \overset{k}{a}]_k]_i \overset{(1)}{=} [x_1^k [y_1^k \overset{k}{a}]_1 [y_1^k \overset{k}{a}]_2 \ldots [y_1^k \overset{k}{a}]_k]_i =$$
$$= [x_1^k y_1^k \overset{k}{a}]_i \overset{(1)}{=} [x_1^k \overset{k}{a}]_i \overset{(2)}{=} [x_i \overset{2k-1}{a}]_1.$$

So, $(L; [\,])$ is a left-zero $(2k, k)$ – semigroup.

(B) Let $D = \{[\overset{2k-1}{a} x]_k \mid x \in Q\}$ and $[a^{2k-1} x_i]_k, [\overset{2k-1}{a} y_i]_k \in D$, $i \in \mathbf{N}_k$. Then:

$$[[\overset{2k-1}{a} x_1]_k \ldots [\overset{2k-1}{a} x_k]_k [\overset{2k-1}{a} y_1]_k \ldots [\overset{2k-1}{a} y_k]_k]_i \overset{(2)}{=}$$
$$= [[\overset{k}{a} x_1^k]_1 [\overset{k}{a} x_1^k]_2 \ldots [\overset{k}{a} x_1^k]_k [\overset{k}{a} y_1^k]_1 [\overset{k}{a} y_1^k]_2 \ldots [\overset{k}{a} y_1^k]_k]_i =$$
$$= [\overset{k}{a} x_1^k [\overset{k}{a} y_1^k]_1 [\overset{k}{a} y_1^k]_2 \ldots [\overset{k}{a} y_1^k]_k]_i \overset{(1)}{=} [\overset{k}{a} [\overset{k}{a} y_1^k]_1 [\overset{k}{a} y_1^k]_2 \ldots [\overset{k}{a} y_1^k]_k]_i =$$
$$= [\overset{k}{a}\overset{k}{a} y_1^k]_i \overset{(1)}{=} [\overset{k}{a} y_1^k]_i \overset{(2)}{=} [\overset{2k-1}{a} y_i]_k.$$

So, $(D; [\,])$ is a right-zero $(2k, k)$ – semigroup.

(C) We define a map $\varphi : L \times D \to Q$ with:

$$(\forall ([x \overset{2k-1}{a}]_1, [\overset{2k-1}{a} y]_k) \in L \times D) \;\; \varphi([x \overset{2k-1}{a}]_1, [\overset{2k-1}{a} y]_k) = [x \overset{k-1}{a} y \overset{k-1}{a}]_1.$$

(C1) We will prove that φ is a well-defined map. Let

$$[x \overset{2k-1}{a}]_1 = [u \overset{2k-1}{a}]_1, [\overset{2k-1}{a}\, y]_k = [\overset{2k-1}{a}\, v]_k.$$

Then:

$$[[x \overset{2k-1}{a}]_1[x \overset{2k-1}{a}]_2 \ldots [x \overset{2k-1}{a}]_k \overset{k}{x}]_1 = [[u \overset{2k-1}{a}]_1[x \overset{2k-1}{a}]_2 \ldots [x \overset{2k-1}{a}]_k \overset{k}{x}]_1$$

$$[x \overset{2k-1}{a}\overset{k}{x}]_1 = [[u \overset{2k-1}{a}]_1[u \overset{2k-1}{a}]_2 \ldots [u \overset{2k-1}{a}]_k \overset{k}{x}]_1$$

$$[x \overset{k-1}{a}\overset{k}{x}]_1 = [u \overset{2k-1}{a}\overset{k}{x}]_1$$

$$[x \overset{k-1}{x}\overset{k}{x}]_1 = [u \overset{k-1}{a}\overset{k}{x}]_1.$$

So, $x = [u \overset{k-1}{a}\overset{k}{x}]_1$.

$$[\overset{k}{y}\, [\overset{2k-1}{a}\, y]_1 \ldots [\overset{2k-1}{a}\, y]_{k-1}[\overset{2k-1}{a}\, y]_k]_k = [\overset{k}{y}\, [\overset{2k-1}{a}\, y]_1 \ldots [\overset{2k-1}{a}\, y]_{k-1}[\overset{2k-1}{a}\, \nu]_k]_k$$

$$[\overset{k}{y}\overset{2k-1}{a}\, y]_k = [\overset{k}{y}\, [\overset{2k-1}{a}\, \nu]_1 \ldots [\overset{2k-1}{a}\, \nu]_{k-1}[\overset{2k-1}{a}\, \nu]_k]_k$$

$$[\overset{k}{y}\overset{k-1}{a}\, y]_k = [\overset{k}{y}\overset{2k-1}{a}\, \nu]_k$$

$$[\overset{k}{y}\overset{k-1}{y}\, y]_k = [\overset{k}{y}\overset{k-1}{a}\, \nu]_k\,.$$

So, $y = [\overset{k}{y}\overset{k-1}{a}\, \nu]_k$. Then:

$$[x \overset{k-1}{a}\, y \overset{k-1}{a}]_1 = [[u \overset{k-1}{a}\overset{k}{x}]_1 \overset{k-1}{a}\, [\overset{k}{y}\overset{k-1}{a}\, \nu]_k \overset{k-1}{a}]_1 \overset{(2)}{=}$$

$$= [[u \overset{k-1}{a}\overset{k}{x}]_1[u \overset{k-1}{a}\overset{k}{x}]_2 \ldots [u \overset{k-1}{a}\overset{k}{x}]_k[\overset{k}{y}\overset{k-1}{a}\, \nu]_k \overset{k-1}{a}]_1 =$$

$$= [u \overset{k-1}{a}\overset{k}{x}\, [\overset{k}{y}\overset{k-1}{a}\, \nu]_k \overset{k-1}{a}]_1 \overset{(1)}{=} [u \overset{k-1}{a}\, [\overset{k}{y}\overset{k-1}{a}\, \nu]_k \overset{k-1}{a}]_1 \overset{(2)}{=}$$

$$= [u \overset{k-1}{a}\, [\overset{k}{y}\, \nu \overset{k-1}{a}]_1[\overset{k}{y}\, \nu \overset{k-1}{a}]_2 \ldots [\overset{k}{y}\, \nu \overset{k-1}{a}]_k]_1 =$$

$$= [u \overset{k-1}{a}\overset{k}{y}\, \nu \overset{k-1}{a}]_1 = [u \overset{k-1}{a}\, \nu \overset{k-1}{a}]_1\,.$$

(C2) We will prove that φ is an injection. Let

$$\varphi([x \overset{2k-1}{a}]_1, [\overset{2k-1}{a}\, y]_k) = \varphi[u \overset{2k-1}{a}]_1, [\overset{2k-1}{a}\, \nu]_k),$$

i.e.

$$[x \overset{k-1}{a}\, y \overset{k-1}{a}]_1 = [u \overset{k-1}{a}\, \nu \overset{k-1}{a}]_1.$$

Then:

$$
\begin{aligned}
&[[x \overset{k-1}{a} y \overset{k-1}{a}]_1[x \overset{k-1}{a} y \overset{k-1}{a}]_2 \ldots [x \overset{k-1}{a} y \overset{k-1}{a}]_k \overset{k}{x}]_1 = \\
&= [[u \overset{k-1}{a} \nu \overset{k-1}{a}]_1[x \overset{k-1}{a} y \overset{k-1}{a}]_2 \ldots [x \overset{k-1}{a} y \overset{k-1}{a}]_k \overset{k}{x}]_1 \\
&[x \overset{k-1}{a} y \overset{k-1}{a} \overset{k}{x}]_1 = [[u \overset{k-1}{a} \nu \overset{k-1}{a}]_1[u \overset{k-1}{a} \nu \overset{k-1}{a}]_2 \ldots [u \overset{k-1}{a} \nu \overset{k-1}{a}]_k \overset{k}{x}]_1 \\
&[x \overset{k-1}{a} \overset{k}{x}]_1 = [u \overset{k-1}{a} \nu \overset{k-1}{a} \overset{k}{x}]_1 \\
&[x \overset{k-1}{x} \overset{k}{x}]_1 = [u \overset{k-1}{a} \overset{k}{x}]_1
\end{aligned}
$$

So, $x = [u \overset{k-1}{a} \overset{k}{x}]_1$.
Similary:

$$
\begin{aligned}
&[\overset{k}{y} [\overset{k-1}{a} x \overset{k-1}{a} y]_1 \ldots [\overset{k-1}{a} x \overset{k-1}{a} y]_{k-1}[x \overset{k-1}{a} y \overset{k-1}{a}]_1]_k = \\
&= [\overset{k}{y} [\overset{k-1}{a} x \overset{k-1}{a} y]_1 \ldots [\overset{k-1}{a} x \overset{k-1}{a} y]_{k-1}[u \overset{k-1}{a} \nu \overset{k-1}{a}]_1]_k \\
&[\overset{k}{y} [\overset{k-1}{a} x \overset{k-1}{a} y]_1 \ldots [\overset{k-1}{a} x \overset{k-1}{a} y]_{k-1}[\overset{k-1}{a} x \overset{k-1}{a} y]_k]_k = \\
&= [\overset{k}{y} [\overset{k-1}{a} u \overset{k-1}{a} \nu]_1 \ldots [\overset{k-1}{a} u \overset{k-1}{a} \nu]_{k-1}[\overset{k-1}{a} u \overset{k-1}{a} \nu]_k]_k \\
&[\overset{k}{y} \overset{k-1}{a} x \overset{k-1}{a} y]_k = [\overset{k}{y} \overset{k-1}{a} u \overset{k-1}{a} \nu]_k \\
&[\overset{k}{y} \overset{k-1}{a} y]_k = [\overset{k}{y} \overset{k-1}{a} \nu]_k \\
&[\overset{k}{y} \overset{k-1}{y} y]_k = [\overset{k}{y} \overset{k-1}{a} \nu]_k.
\end{aligned}
$$

So, $y = [\overset{k}{y} \overset{k-1}{a} \nu]_k$. Then

$$
\begin{aligned}
[x \overset{2k-1}{a}]_1 &= [[u \overset{k-1}{a} \overset{k}{x}]_1 \overset{2k-1}{a}]_1 \overset{(2)}{=} [[u \overset{k-1}{a} \overset{k}{x}]_1 \ldots [u \overset{k-1}{a} \overset{k}{x}]_k \overset{k}{a}]_1 = \\
&= [u \overset{k-1}{a} \overset{k}{x} \overset{k}{a}]_1 \overset{(1)}{=} [u \overset{2k-1}{a}]_1
\end{aligned}
$$

and

$$
\begin{aligned}
[\overset{2k-1}{a} y]_k &= [\overset{2k-1}{a} [\overset{k}{y} \overset{k-1}{a} \nu]_k]_k \overset{(2)}{=} [\overset{k}{a} [\overset{k}{y} \overset{k-1}{a} \nu]_1 \ldots [\overset{k}{y} \overset{k-1}{a} \nu]_k]_k = \\
&= [\overset{k}{a} \overset{k}{y} \overset{k-1}{a} \nu]_k \overset{(1)}{=} [\overset{2k-1}{a} \nu]_k.
\end{aligned}
$$

So $([x \overset{2k-1}{a}]_1, [\overset{2k-1}{a} y]_k) = ([u \overset{2k-1}{a}]_1, [\overset{2k-1}{a} \nu]_k)$, i.e. φ is an injection.
(C3) We will prove that φ is a surjection. Let $x \in Q$. Then

$$
x \overset{(3)}{=} [\overset{2k}{x}]_1 \overset{(2)}{=} [x \overset{k-1}{a} x \overset{k-1}{a}]_1, \quad [x \overset{2k-1}{a}]_1 \in L, \quad [\overset{2k-1}{a} x]_k \in D
$$

and

$$\varphi([x\overset{2k-1}{a}]_1,[\overset{2k-1}{a}x]_k)=[x\overset{k-1}{a}x\overset{k-1}{a}]_1=x.$$

So, φ is a surjection.

(C4) We will prove that φ is a $(2k,k)$ – homomorphism. Let

$$\alpha_i=([x_i\overset{2k-1}{a}]_1,[\overset{2k-1}{a}y_i]_k),\beta_i=([u_i\overset{2k-1}{a}]_1,[\overset{2k-1}{a}\nu_i]_k)\in L\times D, i\in\mathbf{N}_k.$$

Then:

$$\begin{aligned}&[([x_1\overset{2k-1}{a}]_1,[\overset{2k-1}{a}y_1]_k)\dots([x_k\overset{2k-1}{a}]_1,[\overset{2k-1}{a}y_k]_k)([u_1\overset{2k-1}{a}]_1,[\overset{2k-1}{a}\nu_1]_k)\dots\\&\quad\dots([u_k\overset{2k-1}{a}]_1,[\overset{2k-1}{a}\nu_k]_k)]_i=\\&=([[x_1\overset{2k-1}{a}]_1\dots[x_k\overset{2k-1}{a}]_1[u_1\overset{2k-1}{a}]_1\dots[u_k\overset{2k-1}{a}]_1]_i,[[\overset{2k-1}{a}y_1]_k\dots\\&\quad\dots[\overset{2k-1}{a}y_k]_k[\overset{2k-1}{a}\nu_1]_k\dots[\overset{2k-1}{a}\nu_k]_k]_i).\end{aligned}$$

We have:

$$\begin{aligned}\varphi([\alpha_1^k\beta_1^k]_i)&=\varphi([[x_1\overset{2k-1}{a}]_1\dots[u_k\overset{2k-1}{a}]_1]_i,[[\overset{2k-1}{a}y_1]_k\dots[\overset{2k-1}{a}\nu_k]_k]_i)\\&=\varphi([x_i\overset{2k-1}{a}]_1,[\overset{2k-1}{a}\nu_i]_k)=[x_i\overset{k-1}{a}\nu_i\overset{k-1}{a}]_1;\end{aligned}$$

$$\begin{aligned}&[\varphi(\alpha_1)\varphi(\alpha_2)\dots\varphi(\alpha_k)\varphi(\beta_1)\varphi(\beta_2)\dots\varphi(\beta_k)]_i=\\&=[[x_1\overset{k-1}{a}y_1\overset{k-1}{a}]_1\dots[x_k\overset{k-1}{a}y_k\overset{k-1}{a}]_1[u_1\overset{k-1}{a}\nu_1\overset{k-1}{a}]_1\dots[u_k\overset{k-1}{a}\nu_k\overset{k-1}{a}]_1]_i\overset{(2)}{=}\\&=[[x_1^ky_1^k]_1\dots[x_1^ky_1^k]_k[u_1^k\nu_1^k]_1\dots[u_1^k\nu_1^k]_k]_i=\\&=[x_1^ky_1^k[u_1^k\nu_1^k]_1\dots[u_1^k\nu_1^k]_k]_i\overset{(1)}{=}[x_i^ku_1^k\nu_1^k]_i\overset{(1)}{=}[x_1^k\nu_1^k]_i\overset{(2)}{=}[x_i\overset{k-1}{a}\nu_i\overset{k-1}{a}]_1.\end{aligned}$$

So, $\varphi([\alpha_1^k\beta_1^k]_i)=[\varphi(\alpha_1)\varphi(\alpha_2)\dots\varphi(\alpha_k)\varphi(\beta_1)\varphi(\beta_2)\dots\varphi(\beta_k)]_i$, i.e. φ is $(2k,k)$ – homomorphism.

Hence, $\mathbf{Q}$ is a direct product of a left-zero $(2k,k)$ – semigroup and a right-zero $(2k,k)$ – semigroup.

Conversely, let $\mathbf{Q}$ be a direct product of a left-zero $(2k,k)$ – semigroup and a right-zero $(2k,k)$ – semigroup.

(D) Let $(x_i,y_i),(a_i,b_i),(u_i,\nu_i)\in Q$, $i\in\mathbf{N}_k$. Then:

$$\begin{aligned}&[(x_1,y_1)\dots(x_k,y_k)(a_1,b_1)\dots(a_k,b_k)(u_1,\nu_1)\dots(u_k,\nu_k)]\\&=[(x_1,b_1)\dots(x_k,b_k)(u_1,\nu_1)\dots(u_k,\nu_k)]\\&=(x_1,\nu_1)\dots(x_k,\nu_k)\\&=[(x_1,y_1)\dots(x_k,y_k)(u_1,\nu_1)\dots(u_k,\nu_k)].\end{aligned}$$

Hence, $\mathbf{Q}$ satisfies (1).

(E) Let $(x_l, y_l), (a_l, b_l), (u_l, \nu_l), (z_l, w_l) \in Q, l, i, j \in \mathbf{N}_k, (u_j, \nu_j) = (x_i, y_i), (z_j, w_j) = (a_i.b_i)$.

Then:

$$\begin{aligned}
&[(x_1, y_1) \dots (x_i, y_i) \dots (x_k, y_k)(a_1, b_1) \dots (a_i, b_i) \dots (a_k, b_k)]_i = \\
&= (x_i, b_i) = \\
&= [(u_1, \nu_1) \dots (u_{j-1}, \nu_{j-1})(x_i, y_i) \dots \\
&\dots (u_k, \nu_k)(z_1, w_1) \dots (z_{j-1}, w_{j-1})(a_i, b_i) \dots (z_k, w_k)]_j.
\end{aligned}$$

Hence, $\mathbf{Q}$ satisfies (2).

(F) Let $(a, b) \in Q$. Then $[\overset{2k}{(a, b)}] = \overset{k}{(a, b)}$. Hence, $\mathbf{Q}$ satisfies (3). □

Proposition 2. Let $\mathbf{Q} = (Q; [\,])$ be a $(2k, k)$ – semigroup. Then $\mathbf{Q}$ is a direct product of a left-zero $(2k, k)$ – semigroup and a right-zero $(2k, k)$ – semigroup if and only if there exist semigroup $(Q; *)$ which is a rectangular band, i.e. a direct product of a left-zero semigroup and a right-zero semigroup, such as

$$[x_1^k y_1^k]_i = x_i * y_i, \; x_1^k, y_1^k \in \mathbf{Q}^k, \quad i \in \mathbf{N}_k.$$

Proof. Suppose $\mathbf{Q} = (Q; [\,])$ is a $(2k, k)$ – semigroup, direct product of a left-zero $(2k, k)$ – semigroup and a right-zero $(2k, k)$ – semigroup. According to Proposition 1. we have:

(1) $[abc] = [ac]$, $a, b, c \in Q^k$;

(2) $[a_1^{i-1} a a_{i+1}^k b_1^{i-1} b b_{i+1}^k]_i = [x_1^{j-1} a x_{j+1}^k y_1^{j-1} b y_{j+1}^k]_j$, $i, j \in \mathbf{N}_k$;

(3) $[\overset{2k}{a}] = \overset{k}{a}$.

For a fixed $a \in Q$, let $*$ be an operation defined on Q, by

$$x * y = [x \overset{k-1}{a} y \overset{k-1}{a}]_1, \quad x, y \in Q.$$

(A) Clearly $(Q; *)$ is groupoid.

(B) We will prove that $(Q; *)$ is semigroup. Let $x, y, z \in Q$. Then:

$$\begin{aligned}
(x * y) * z &= [[x \overset{k-1}{a} y \overset{k-1}{a}]_1 \overset{k-1}{a} z \overset{k-1}{a}]_1 \overset{(2)}{=} \\
&= [[x \overset{k-1}{a} y \overset{k-1}{a}]_1 \dots [x \overset{k-1}{a} y \overset{k-1}{a}]_k z \overset{k-1}{a}]_1 = \\
&= [x \overset{k-1}{a} y \overset{k-1}{a} z \overset{k-1}{a}]_1 \overset{(1)}{=} [x \overset{k-1}{a} z \overset{k-1}{a}]_1;
\end{aligned}$$

$$\begin{aligned}
x * (y * z) &= [x \overset{k-1}{a} [y \overset{k-1}{a} z \overset{k-1}{a}]_1 \overset{k-1}{a}]_1 \overset{(2)}{=} \\
&= [x \overset{k-1}{a} [y \overset{k-1}{a} z \overset{k-1}{a}]_1 \dots [y \overset{k-1}{a} z \overset{k-1}{a}]_k]_1 = \\
&= [x \overset{k-1}{a} y \overset{k-1}{a} z \overset{k-1}{a}]_1 \overset{(1)}{=} [x \overset{k-1}{a} z \overset{k-1}{a}]_1.
\end{aligned}$$

Hence, $(x * y) * z = x * (y * z)$, i.e. $(Q; *)$ is semigroup.

(C) In $(Q; *)$ we have $x * y * z = [x \overset{k-1}{a} z \overset{k-1}{a}]_1 = x * z$ and

$$x * x = [x \overset{k-1}{a} x \overset{k-1}{a}]_1 \overset{(2)}{=} [x \overset{k-1}{x} x \overset{k-1}{x}]_1 \overset{(3)}{=} x.$$

Hence, $(Q; *)$ is semigroup in which $x * y * z = x * z$ and $x * x = x$, i.e. $(Q; *)$ is a rectangular band.

(D) At the end, for $x_1^k, y_1^k \in Q^k$, $i \in \mathbf{N}_k$ we have

$$[x_1^k y_1^k]_i \overset{(2)}{=} [x_i \overset{k-1}{a} y_i \overset{k-1}{a}]_1 = x_i * y_i.$$

So, $(Q; *)$ is semigroup, direct product of a left-zero semigroup and a right-zero semigroup and $[x_1^k y_1^k]_i = x_i * y_i$.

Conversely, let $(Q; [\,])$ be a $(2k, k)$ – semigroup and there exist semigroup $(Q; *)$ which is a rectangular band, such that

$$[x_1^k y_1^k]_i = x_i * y_i, x_1^k, y_1^k \in Q^k, \quad i \in \mathbf{N}_k.$$

Then in $(Q; *)$ holds:

(a) $x * y * z = x * z$ and

(b) $x * x = x$

We will prove that for $\mathbf{Q}$ the statements (1), (2) and (3) from Proposition 1. are true and with that the proof will be completed.

(E) Let $x_1^k, y_1^k, z_1^k \in Q^k$. Then:

$$[x_1^k y_1^k z_1^k] = [[x_1^k y_1^k] z_1^k] = [x_1 * y_1, x_2 * y_2, \ldots, x_k * y_k, z_1, \ldots, z_k] =$$
$$= (x_1 * y_1 * z_1, x_2 * y_2 * z_2, \ldots x_k * y_k * z_k) \overset{(a)}{=}$$
$$= (x_1 * z_1, x_2 * z_2, \ldots, x_k * z_k) = [x_1^k z_1^k].$$

So, the statement (1) from Proposition 1. is true.

(F) Let $x_1^k, y_1^k, a_1^k, b_1^k \in Q^k$, $i, j \in \mathbf{N}_k$ and $x_i = a_j, y_i = b_j$. Then: $[x_1^{i-1} x_i x_{i+1}^k y_1^{i-1} y_i y_{i+1}^k]_i = x_i * y_i = [a_1^{j-1} x_i a_{j+1}^k b_1^{j-1} y_i b_{j+1}^k]_j$. So, the statement (2) from Proposition 1. is true.

(G) Let $x \in Q$. Then: $[\overset{2k}{x}]_i = x * x \overset{(b)}{=} x$, i.e. $[\overset{2k}{x}] = \overset{k}{x}$. So, the statement (3) from Proposition 1. is true. □

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КАРАКТЕРИЗАЦИЈА НА $(2k, k)$ – ПРАВОАГОЛНА ЛЕНТА

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Резиме

$(2k, k)$–полугрупата $(Q; [\,])$ која е директен производ на една лево нулта $(2k, k)$–полугрупа и една десно нулта $(2k, k)$–полугрупа се нарекува $(2k, k)$–правоаголна лента. Во трудот е дадена карактеризација на $(2k, k)$–правоаголна лента.

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