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UNITS IN VECTOR VALUED GROUPS

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Abstract

In this paper the units of $(m+k,m)$-groups and $com(m+k,m)$-groups are considered. Some properties of them are presented. Units of type $(e,e,\cdots,e)$ in both cases are also studied. In the last section some examples of $com(3,2)$-groups with units are given.

1. Preliminaries

Vector valued $(m+k,m)$-groupoids, semigroups and groups were introduced in [16] and [1]. These algebraic structures generalize the usual binary $(2,1)$-algebraic structures, and also they have new ideas and specific properties.

For a set S and a positive integer q, S^q denotes the q-th Cartesian power of S. We use the notation x_1^q instead of $x=(x_1,\cdots,x_q)$ for elements $x\in S^q$. Let n,m,k be positive integers and $n=m+k$.

An $(m+k,m)$-*groupoid* is a pair (G,f) where G is a nonempty set, f is an $(m+k,m)$-*operation*, i.e. a map $f:G^{m+k}\to G^m$. The notions of semigroups and groups are generalized to the notions of $(m+k,m)$-semigroups and $(m+k,m)$-groups. An $(m+k,m)$-groupoid (G,f) is called $(m+k,m)$-*semigroup* if for each $x_1^{m+2k}\in G^{m+2k}$, and each $i:\ 1\le i\le k$

$$f(x_1^if(x_{i+1}^{i+n})x_{i+n+1}^{n+k})=f(f(x_1^n)x_{n+1}^{n+k}). \tag{1.1}$$

An $(m+k,m)$-semigroup (G,f) is called $(m+k,m)$-*group* if for each $a\in G^k$, $b\in G^m$, the equations

$$f(xa) = b = f(ay) \tag{1.2}$$

have solutions $x, y \in G^m$. Note that $(m+k, m)$-groups satisfy stronger condition than (1.2). Namely, for any fixed $a_1, \cdots, a_k, b_1, \cdots, b_m \in G$ the equation

$$f(a_1^j x_1^m a_{j+1}^k) = (b_1, \cdots, b_m), \qquad 0 \le j \le k, \tag{1.3}$$

has a unique solution $x_1, \cdots, x_m$ [2], and in section 2 we will use this statement.

For $m = 1 = k$, the above notions are the usual notions of binary groupoids, semigroups and groups, and for $m = 1$, $n = k+1$, they are the notions of n-groupoids, n-semigroups and n-groups. Further on we assume that $m \ge 2$.

The $(m+k, m)$-semigroups and $(m+k, m)$-groups are examined in [1,2,3,7]. There are a lot of natural examples of $(m+k, m)$-semigroups. The description of the free $(m+k, m)$-semigroups is given in [6]. The existence of nontrivial $(m+k, m)$-groups (G, f), depends crucially on k. For $k < m$, there do not exist finite nontrivial $(m+k, m)$-groups. If $k = m$, then $(2m, m)$-groups are closely related to the usual groups, with additional rich structures [3, 10]. For $k > m$, there are a lot of examples of $(m+k, m)$-groups [7]. The description of free $(m+k, m)$-groups is given in [8, 9]. In [11, 13] the nonexistence of some continuous $(m+k, m)$-groups for $k < m$ is proved.

In section 2 we will use the following result. For each $(m+k, m)$ group (G, f) there is a binary group $\mathbf{H}$ such that $G \subset H$ and for each $(x_1, \cdots, x_{m+k}) \in G^{m+k}$ and $(y_1, \cdots, y_m) \in G^m$,

$$f(x_1^{m+k}) = y_1^m \quad \Leftrightarrow \quad x_1 \cdot x_2 \cdots x_{m+k} = y_1 \cdot y_2 \cdots y_m. \tag{1.4}$$

Such an example of group $\mathbf{H}$ is the universal covering group [2]. Namely, if (G, f) is an $(m+k, m)$-group, then (by [2]) the universal covering group is denoted by $\mathbf{G}^\vee$ and $G^\vee = G^m \cup G_{m+1} \cup \ldots \cup G_{m+k-1}$, where $G_{m+j} = G^{m+j}/\beta$ and β is the restriction on G^{m+j} of the congruence relation on G^+ generated by

$\Lambda = \{(A_1^n, b_1^m) | f(a_1^n) = b_1^m \text{ in } G\}$.

Another type of vector valued structures is the theory of fully commutative vector valued structures and now we give a brief overview of it.

Let $S^{(m)}$ be the m-th symmetric product of S, i.e. $S^{(m)} = S^m/\approx$ where $\approx$ is equivalence on S^m defined by $x_1^m \approx y_1^m$ if and only if $x_1, x_2, \cdots, x_m$ is a permutation of $y_1, y_2, \cdots, y_m$.

A map $f : G^{(n)} \to G^{(m)}$ is called a *fully commutative* (n, m)*-operation* on G, and the pair (G, f) is called a *fully commutative* (n, m)*-groupoid*. A fully commutative (n, m)-groupoid is called a *fully commutative* (n, m)*-semigroup*, if for each $1 \le i \le k$ and each $x_1^{n+k} \in G^{(n+k)}$,

$$f(x_1^i f(x_{i+1}^{i+n}) x_{i+n+1}^{n+k}) = f(f(x_1^n) x_{n+1}^{n+k}). \tag{1.5}$$

A fully commutative (n,m)-semigroup (G,f) is called a *fully commutative* (n,m) *-group* if for each $a \in G^{(k)}, b \in G^{(m)}$, the equation

$$f(ax) = b \tag{1.6}$$

has solution $x \in G^{(m)}$.

Further on we will write $com(m+k,m)$-groupoid (semigroup, group) instead of fully commutative $(m+k,m)$-groupoid (semigroup, group). The notion of $com(m+k,m)$-groupoids, semigroups and groups are introduced in [5] and [4]. The free $com(m+k,m)$-groups are described in [4]. Some examples of $com(m+k,m)$-groups are given in [12]. Namely, these examples are constructed over the set of complex numbers $\mathbf{C}$ or subset of $\mathbf{C}$, and they naturally lead to the affine and projective $com(m+k,m)$-groups. These groups are well studied in [14, 15] and we will refer to them in section 4. Moreover, these groups are uniquely known as continuous $com(m+k,m)$-groups.

Note that a $com(m+k,m)$-semigroup (group) induces inductively a unique $com(m+sk,m)$-semigroup (group) for $s \geq 2$, by

$$f(x_1^{m+sk}) = f(f(x_1^{m+(s-1)k})x_{m+(s-1)k+1}^{m+sk}). \tag{1.7}$$

In this paper we consider the units in $(m+k,m)$-groups and also in $com(m+k,m)$-groups, and give some of their properties and some examples. In both cases we study special units of type $(e,e,\ldots,e)$.

2. Units in (m+k,m)-groups

First we introduce the following definition.

Definition 2.1. Let $f : G^{m+k} \to G^m$ satisfy the following conditions:

(i) $f(x_1^i f(x_{i+1}^{i+n})x_{i+n+1}^{n+k}) = f(f(x_1^n)x_{n+1}^{n+k}), \quad 1 \leq i \leq k,$
where $n = m+k$ and $x_1, \cdots, x_{n+k} \in G$,

(ii) there is an element $e \in G$, which will be called *unit*, such that for any $x_1, \cdots, x_m \in G$

$$f(x_1^i, \underbrace{e, \cdots, e}_{k}, x_{i+1}^m) = (x_1, \cdots, x_m), \qquad 0 \leq i \leq m,$$

(iii) for any fixed $a_1, \cdots, a_k, b_1, \cdots, b_m \in G$ the equation

$$f(a_1^j x_1^m a_{j+1}^k) = (b_1, \cdots, b_m), \qquad j \in \{0, k\},$$

has a solution of $x_1, \cdots, x_m$.
Then (G,f) will be called $(m+k,m)$*-group with unit.*

Three cases should be considered here. If $k > m$, then the element e (if such an element exists) such that $(e,e,\cdots,e)$ is a unit may not be unique. This can be seen from the following simple example.

Example 2.1. Let us consider the following (6,2)-group $(R \setminus \{0\}, f)$ defined by

$$f(x_1,x_2,x_3,x_4,x_5,x_6) = (x_1x_3x_5, x_2x_4x_6).$$

Then

$$f(1,1,1,1,x,y) = f(x,1,1,1,1,y) = f(x,y,1,1,1,1) = (x,y)$$

and also

$$f(-1,-1,-1,-1,x,y){=}f(x,-1,-1,-1,-1,y){=}f(x,y,-1,-1,-1,-1){=}(x,y)$$

and hence the unit is not unique.

If $k = m$, then there is a unique element e such that $(e, e, \cdots, e) \in G^m$ is a unit. Namely each $(2m, m)$-group can be considered as a usual binary, i.e. (2,1)-group $(G^m, *)$, and it is known [3] that the corresponding unit is of type $(e, e, \cdots, e)$. satisffied.

Example 2.2. Let $(G, *)$ be a usual binary group. Then

$$f(x_1, x_2, \cdots, x_m, x_{m+1}, x_{m+2}, \cdots, x_{2m}){=}(x_1 * x_{m+1}, x_2 * x_{m+2}, \cdots, x_m * x_{2m})$$

defines a $(2m, m)$-group with a unit element e, where e is the unit of $(G, *)$. Namely, the element e satisfies the condition (ii) of the definition 2.1.

If $k < m$, we prove the following proposition.

Proposition 2.1. *If (G, f) is an $(m + k, m)$-group with unit and $k < m$, then $|G| = 1$.*

Proof. If $k < m$, then according to (ii) we get

$$f(\underbrace{e, e, \cdots, e}_{k}, x_1, x_2, \cdots, x_{m-k}, \underbrace{e, e, \cdots, e}_{k}) = (\underbrace{e, e, \cdots, e}_{k}, x_1, x_2, \cdots, x_{m-k}),$$

and

$$f(\underbrace{e, e, \cdots, e}_{k}, x_1, x_2, \cdots, x_{m-k}, \underbrace{e, e, \cdots, e}_{k}) = (x_1, x_2, \cdots, x_{m-k}, \underbrace{e, e, \cdots, e}_{k}).$$

If $x_1 \neq e$ we obtain a contradiction. Thus $x_1 = e$, and $|G| = 1$. □

The following proposition is obvious.

Proposition 2.2. *Each $(m+k, m)$-group (G, f) with unit induces an $(m + sk, m)$-group with unit.*

Note that if $(e, \cdots, e) \in G^k$ is a unit in (G, f), then $(e, \cdots, e) \in G^{sk}$ is a unit in the induced $(m + sk, m)$-group.

The condition (ii) in definition 2.1 can be replaced by the following weaker condition:

(ii′) There are elements $e, x_1, \cdots, x_m \in G$ and there is an $i \in \{0, \cdots, m\})$, such that

$$f(x_1^i, \underbrace{e, \cdots, e}_{k}, x_{i+1}^m) = (x_1, \cdots, x_m).$$

The proof follows from the proposition 2.3.

Now we will consider the general case of unit $(e_1,\cdots,e_k)$. There are some different ways to define it. First, we can say that $(e_1,\cdots,e_k)$ is a *unit string* in an (m+k,m)-group, if

$$f(x_1,\cdots,x_j,e_1,\cdots,e_k,x_{j+1},\cdots,x_m)=(x_1,\cdots,x_m),$$

for each $x_1,\cdots,x_m\in G$ and each $j: 1\le j\le m$. In this case, analogously to proposition 2.1, it is easy to verify that $|G|=1$ if $k<m$. In order to avoid such a situation we can require a unit string in the induced $(m+sk,m)$-group and thus the unit string should belong to G^{sk}.

According to the previous discussion we assume that $k\ge m$ and we will describe all the unit strings in an $(m+k,m)$-group. First we prove the following:

Proposition 2.3. *Let $k\ge m$. Assume that there are elements $e_1,\cdots,e_k$, $x_1,\cdots,x_m\in G$ and there is a $j\in\{0,1,\cdots,k-m\}$ such that*

$$f(x_1^je_1^kx_{j+1}^m)=(x_1,\cdots,x_m).$$

Then $(e_1,\cdots,e_k)$ is a unit string, i.e. the previous equality is true for arbitrary $x_1,\cdots,x_m$ and arbitrary j $(0\le j\le k-m)$.

Proof. Let $H=G^\wedge$ be the universal covering group of **G**. If $f(x_1^je_1^kx_{j+1}^m)=(x_1,\cdots,x_m)$, then according to (1.4),

$$x_1\cdots x_j\cdot e_1\cdots e_k\cdot x_{j+1}\cdots x_m=x_1\cdots x_m$$

is true in H. Hence $e_1\cdots e_k$ is a unit in H and thus

$$x_1\cdots x_j\cdot e_1\cdots e_k\cdot x_{j+1}\cdots x_m=x_1\cdots x_m$$

is true for arbitrary $x_1,\cdots,x_m$ and arbitrary j $(0\le j\le k-m)$. Thus

$$f(x_1^je_1^kx_{j+1}^m)=(x_1,\cdots,x_m),$$

is true for arbitrary $x_1,\cdots,x_m$ and arbitrary j. □

According to proposition 2.3, the definition of a unit string $(e_1,\cdots,e_k)$ reduces for example only to a left unit string, and moreover the following equality

$$f(e_1^kx_1^m)=(x_1,\cdots,x_m)\tag{2.1}$$

is sufficient to be true for one m-tuple $(x_1,\cdots,x_m)$. This result does not tell us anything about the existence of a unit string $(e_1,\cdots,e_k)$. If $k=m$ the unit string is unique. If $k>m$, the set of unit strings is of the same cardinality as G^{k-m}. Namely, for any elements $x_1,\cdots,x_{k-m}$ in G, the equation (2.1) has a unique solution on $x_{m+1},\cdots,x_k$. Note that for arbitrary $x_1,\cdots,x_j,x_{j+m+1},\cdots,x_k$, the equation (2.1) has also a unique solution on $x_{j+1},\cdots,x_{j+m}$, $(0\le j\le k-m)$.

The next property gives an additional property of a unit string $(e_1,\cdots,e_k)$.

Proposition 2.4. *If $k > m$ and $(e_1, \cdots, e_k)$ is a unit string in an $(m+k, m)$-group (G, f), then $(e_i, \cdots, e_k, e_1, \cdots, e_{i-1})$ is a unit string for each i, as well.*

Proof. It is sufficient to prove that if $(e_1, e_2, \cdots, e_k)$ is a unit string, then $(e_k, e_1, e_2, \cdots, e_{k-1})$ is also a unit string. Since $(e_1, \cdots, e_k)$ is a unit string, we obtain

$$f(e_k, e_1, e_2, \cdots, e_{k-1}, e_k, x_1, \cdots, x_{m-1}) = (e_k, x_1, \cdots, x_{m-1})$$

for any $x_1, \cdots, x_{m-1}$. Now according to the proposition 2.3 we conclude that $(e_k, e_1, e_2, \cdots, e_{k-1})$ is a unit string as well. □

3. Units in com(m+k,m)-groups

We consider an analogue to the definition 2.1 for fully commutative $(m+k, m)$-groups. Instead of G^i we consider the permutation product $G^{(i)}$.

Definition 3.1. Let $f : G^{(m+k)} \to G^{(m)}$ satisfy the following conditions:

(i) $f(x_1^i f(x_{i+1}^{i+n}) x_{i+n+1}^{n+k}) = f(f(x_1^n) x_{n+1}^{n+k}), \quad 1 \le i \le k,$
where $n = m + k$ and $x_1, \cdots, x_{n+k} \in G$,

(ii) there is an element $e \in G$, which will be called *unit*, such that for any $x_1, \cdots, x_m \in G$

$$f(x_1^m, \underbrace{e, \cdots, e}_{k}) = (x_1, \cdots, x_m),$$

(iii) for any fixed $a_1, \cdots, a_k, b_1, \cdots, b_m \in G$ the equation

$$f(a_1^k x_1^m) = (b_1, \cdots, b_m),$$

has a solution on $x_1, \cdots, x_m$ (unique up to permutation).
Then (G, f) will be called *com$(m+k, m)$-group with unit.*

Analogously as in section 2, the condition (ii) in definition 3.1 can be replaced by the following weaker condition:

(ii′) There are $e, x_1, \cdots, x_m \in G$,

$$f(x_1^m, \underbrace{e, \cdots, e}_{k}) = (x_1, \cdots, x_m).$$

Proposition 3.1. *If $k \le m$ and (G, f) is a com$(m+k, m)$-group with unit $(e, \cdots, e)$, then the element e is unique.*

Proof. Assume that $k \le m$ and (G, f) is a *com*$(m+k, m)$-group with special units $(e, \cdots, e)$ and $(e', \cdots, e')$. For $k = m$:

$$(\underbrace{e, \cdots, e}_{m}) = f(\underbrace{e, \cdots, e}_{m}, \underbrace{e', \cdots, e'}_{m}) = (\underbrace{e', \cdots, e'}_{m}),$$

and thus $e = e'$. If $k < m$, then

$$f(x_1^{m-k}, \underbrace{e, \cdots, e}_{k}, \underbrace{e', \cdots, e'}_{k}) = (x_1^{m-k}, \underbrace{e, \cdots, e}_{k})$$

and

$$f(x_1^{m-k}, \underbrace{e, \cdots, e}_{k}, \underbrace{e', \cdots, e'}_{k}) = (x_1^{m-k}, \underbrace{e', \cdots, e'}_{k})$$

are true for arbitrary $x_1, \cdots, x_{m-k}$ and this is possible only if $e = e'$. □

Note that if $k > m$ and (G, f) is a $com(m + k, m)$-group with a unit $(e, \cdots, e)$, then the element e may not be unique. It can be seen from the following example.

Example 3.1. Let us consider the following $com(6, 2)$-group on $C \setminus \{0, 1\}$ defined by

$$f(z_1, z_2, z_3, z_4, z_5, z_6) = (w_1, w_2) \quad \Leftrightarrow$$

$$\begin{cases} z_1 z_2 z_3 z_4 z_5 z_6 = w_1 w_2 \\ (1 - z_1)(1 - z_2)(1 - z_3)(1 - z_4)(1 - z_5)(1 - z_6) = -4(1 - w_1)(1 - w_2) \end{cases}.$$

Note that (i, i, i, i) and $(-i, -i, -i, -i)$ are two units of the above $com(6, 2)$-group.

Proposition 3.2. *Each $com(m + k, m)$-group with unit induces a $com(m + sk, m)$-group with unit.*

Proposition 3.3. *Each $com(m + k, m)$-group with unit is induced via a $com(m + 1, m)$-group with unit.*

Proof. Assume that $f : G^{(m+k)} \to G^{(m)}$ defines a $com(m + k, m)$-group (G, f) with a unit e. Let $g : G^{(m+1)} \to G^{(m)}$ be defined by

$$g(x_1^{m+1}) = f(x_1^{m+1}, e, \cdots, e).$$

Then it is easy to verify that (G, g) is a $com(m + 1, m)$-group with unit, and moreover it induces the given $com(m + k, m)$-group with unit. □

According to proposition 3.3 it is sufficient to consider only $com(m + 1, m)$-groups with unit. From the definition 3.1 it follows that a $com(m+1, m)$-group (G, f) is also a $com(m+1, m)$-group with unit iff there is an element $e \in G$ such that $f(x_1^m e) = (x_1^m)$. Thus we are able to find some examples of $com(m + 1, m)$-groups with units, and hence $com(m + k, m)$-groups with units. Such examples are given in section 4.

We will prove bellow some properties concerning homomorphisms between fully commutative vector valued groups with units.

Proposition 3.4. *Let* **G** *be a $com(m + k, m)$-group with a unit e and* **G**$'$ *a $com(m + k, m)$-group. If there is a homomorphism $\varphi : G \to G'$, then $\varphi(e) = e'$ is a unit in* **G**$'$.

Proof. $\varphi(e) = e'$ is a unit in $\varphi(G)$. Let p be the least positive integer such that $m + p \equiv 0(mod\ k)$ and $1 \in G'_{m+p}$ is the unit in the universal covering group $G'^{\wedge}$. Then, for each $x' \in \varphi(G)^{(m)}$ we have $1x' = x' = \varphi(e)^k x' \Rightarrow 1 = \varphi(e)^k$, i.e. $\varphi(e) = e'$ is a unit in **G**". □

Proposition 3.5. *Let* **G** *and* **G**$'$ *be com*$(m+k, m)$*-groups with units* e *and* e', *respectively and* $\varphi : G \to G'$ *a homomorphism, such that* $\varphi(e) = e'$. *Then*

$$H := \{x|\varphi(x) = e'\}$$

is a com$(m + k, m)$*-subgroup of* **G** *with a unit* e.

Proof. It is clear that $e \in H$. Let $x_1, \dots, x_{m+k} \in H$. If $f(x_1^{m+k}) = y_1^{m+k}$, then $f'(\varphi(x_1)\dots\varphi(x_{m+k})) = f'(e'^{m+k}) = e'^m$. As φ is a homomorphism, then $\varphi(y_1)\dots\varphi(y_m) = e'^m \Rightarrow \varphi(y_i) = e'$, for each $i \in \{1, \dots, m\}$, i.e. $y_i \in H$.

Thus, H is a com $(m + k, m)$-subsemigroup of **G** with a unit e. Let $a_1, \dots, a_k, b_1, \dots, b_m \in H$ and $x_1, \dots, x_m \in G$ are such that $f(a_1^k x_1^m) = b_1^m$. As $\varphi(a_i) = e'$, $\varphi(b_j) = e'$, we have

$$f'(\varphi(a)_1^k \varphi(x)_1^m) = \varphi(b_1)\dots\varphi(b_m),$$

i.e.

$$f'(e'^k \varphi(x)_1^m) = e'^m.$$

As e' is a unit in **G**$'$, we have $f'(e'^k\varphi(x)_1^m) = \varphi(x_1)\dots\varphi(x_m)$. Thus $\varphi(x_i) = e'$, i.e. $x_i \in H$ for all $i \in \{1, \dots, m\}$. □

Analogously as in section 2, we say that $(e_1, \cdots, e_k)$ is a *unit string* of a *com*$(m + k, m)$-group (G, f) if

$$f(x_1, \cdots, x_m, e_1, \cdots, e_k) = (x_1, \cdots, x_m)$$

for each $x_1, \cdots, x_m \in G$. In fact, it is sufficient that the previous equality is true for one m-tuple $(x_1, \cdots, x_m)$. Moreover, if $k > m$ and $e_{i_1}, \cdots, e_{i_{k-m}}$ are fixed, then e_j for $1 \le j \le k$ and $j \ne i_s$ for $s = 1, \cdots, k-m$, are uniquely determined up to permutation.

4. Some examples of fully commutative vector valued groups with units

We refer to the notation and results in [14] and [15]. Let us consider the special case $m = 2$ and $k = 1$. We denote by t the number of excluded points from the complex plane. It is proved in [14, 15] that up to isomorphism there are only two affine $com(3, 2)$-groups (and hence $com(2+k, 2)$-groups) on **C** ($t = 0$), and they are given by

$$f(z_1^3) = w_1^2 \quad \Leftrightarrow \quad \begin{cases} z_1 + z_2 + z_3 = w_1 + w_2 \\ z_1z_2 + z_2z_3 + z_3z_1 = w_1w_2 \end{cases}, \tag{4.1}$$

$$f(z_1^3) = w_1^2 \quad \Leftrightarrow \quad \begin{cases} z_1 + z_2 + z_3 = w_1 + w_2 \\ z_1 z_2 + z_2 z_3 + z_3 z_1 = 1 + w_1 w_2 \end{cases} . \qquad (4.2)$$

If $t = 1$, then it can be proved that each affine $com(3,2)$-group is isomorphic to one of the $com(3,2)$-groups defined on $\mathbf{C} \setminus \{0\}$

$$f(z_1^3) = w_1^2 \quad \Leftrightarrow \quad \begin{cases} z_1 \cdot z_2 \cdot z_3 = w_1 \cdot w_2 \\ z_1 + z_2 + z_3 = \lambda + w_1 + w_2 \end{cases} , \quad (\lambda \in \mathbf{C}). \qquad (4.3)$$

The parameter λ in (4.3) can not vanish or be changed to a constant, for example $\lambda = 0$ or $\lambda = 1$, but note that λ and $\overline{\lambda}$ give isomorphic groups. If $t = 2$, then without loss of generality we can assume that the singular points are 0 and 1 (and ∞ too). So each affine $com(3,2)$-group on $\mathbf{C}\setminus\{\alpha, \beta\}$ $(\alpha \neq \beta)$ is isomorphic to one of the $com(3,2)$-groups defined on $\mathbf{C} \setminus \{0, 1\}$

$$f(z_1^3) = w_1^2 \quad \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z_1 \cdot z_2 \cdot z_3 = \lambda \cdot w_1 \cdot w_2 \\ (z_1 - 1)(z_2 - 1)(z_3 - 1) = \mu(w_1 - 1)(w_2 - 1) \end{cases} \quad (\lambda, \mu \in \mathbf{C} \setminus \{0\}) \qquad (4.4)$$

and some of the groups in (4.4) are isomorphic, but the parameters λ and μ can not vanish.

Now it is easy to see which of the above $com(3,2)$-groups are also $com(3,2)$-groups with units.

If $t = 0$, then the $com(3,2)$-group defined by (4.1) is also a $com(3,2)$-group with unit, while the $com(3,2)$-group defined by (4.2) is not. It $t = 1$, then the $com(3,2)$-group defined by (4.3) is a $com(3,2)$-group with unit if and only if $\lambda = 1$. Thus, in this case (up to isomorphism) we have only one $com(3,2)$-group

$$f(z_1^3) = w_1^2 \quad \Leftrightarrow \quad \begin{cases} z_1 \cdot z_2 \cdot z_3 = w_1 \cdot w_2 \\ z_1 + z_2 + z_3 = 1 + w_1 + w_2 \end{cases} . \qquad (4.5)$$

If $t = 2$, then the $com(3,2)$-group defined by (4.4) is a $com(3,2)$-group with unit if and only if $\mu = \lambda - 1$. Thus, in this case we have the following parametric set of $com(3,2)$-groups with units

$$f(z_1^3) = w_1^2 \quad \Leftrightarrow \qquad (4.6)$$

$$\begin{cases} z_1 \cdot z_2 \cdot z_3 = \lambda \cdot w_1 \cdot w_2 \\ (z_1 - 1)(z_2 - 1)(z_3 - 1) = (\lambda - 1)(w_1 - 1)(w_2 - 1) \end{cases} \quad (\lambda \in \mathbf{C} \setminus \{0, 1\}).$$

Thus for the affine $com(3,2)$-groups with units we obtain the group defined by (4.1), (4.5) and (4.6). Moreover, according to the theorem 5.3 [15], the groups defined by (4.1) and (4.4) are not isomorphic and no one of them is isomorphic to any group from (4.6). In the last step we will classify the groups in (4.6) up to isomorphism. Namely, according to the corollary 5.2 [15], any isomorphism between affine or projective $com(m+k, m)$-groups is induced by a complex map having one of the following forms:

$$\varphi(z) = \frac{az+b}{cz+d} \quad \text{or} \quad \varphi(z) = \frac{a\bar{z}+b}{c\bar{z}+d} \quad (ad-bc \neq 0).$$

Moreover such a map $\varphi(z)$ must send the set $\{1, 0, \infty\}$ into itself because $1, 0, \infty$ are singular elements for the group (4.6) and any isomorphism must send any singular point into singular point preserving the multiplicity of the singular point (see theorem 5.3 [15]). Thus any such isomorphism is induced by a map which belongs to the following group of transformations:

$$H = \{z \mapsto 1-z,\ z \mapsto \frac{1}{1-z},\ z \mapsto \frac{1}{z},\ z \mapsto 1-\frac{1}{z}, z \mapsto 1-\frac{1}{1-z},$$

$$z \mapsto 1-\bar{z},\ z \mapsto \frac{1}{1-\bar{z}},\ z \mapsto \frac{1}{\bar{z}},\ z \mapsto 1-\frac{1}{\bar{z}},\ z \mapsto 1-\frac{1}{1-\bar{z}}\}. \qquad (4.7)$$

Further, each φ of these ten mappings identifies λ with $\varphi(\lambda)$. Moreover, $\lambda \in \mathbf{C} \setminus \{0, 1\}$ and the group H acts over the set $\mathbf{C} \setminus \{0, 1\}$ and hence any two orbits of λ from (4.6) yield to non-isomorphic $com(3, 2)$-groups with units. Hence we studied the affine $com(3, 2)$-groups with units up to isomorphism.

In the general case, suppose that an affine $com(m+1, m)$-group is given via a matrix $A = (\alpha_{ij})$. Then it is easy to see whether this affine $com(m+1, m)$-group is a $com(m+1, m)$-group with unit or not. Namely, it is a $com(m+1, m)$-group with unit if and only if there exists an element $z \in \mathbf{C}$ such that $\bar{\varphi}_z$ is the unit $(m+1)\times(m+1)$ matrix. This transformation is given by the following affine transformation

$$\bar{\varphi}_z = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \alpha_{10}+z\alpha_{11} & \alpha_{11}+z\alpha_{12} & \alpha_{12}+z\alpha_{13} & \cdots & \alpha_{1m}+z\alpha_{1,m+1} \\ \vdots & & & & \\ \alpha_{m0}+z\alpha_{m1} & \alpha_{m1}+z\alpha_{m2} & \alpha_{m2}+z\alpha_{m3} & \cdots & \alpha_{mm}+z\alpha_{m,m+1} \end{bmatrix}.$$

This $com(m+1, m)$-group has the following representation

$$f(z_1^{m+1}) = w_1^m \quad \Leftrightarrow \quad \overline{\varphi}_{z_1} \cdot \overline{\varphi}_{z_2} \cdots \overline{\varphi}_{z_{m+1}} = \overline{\varphi}_{w_1} \cdot \overline{\varphi}_{w_2} \cdots \overline{\varphi}_{w_m}.$$

References

[1] G.Čupona: *Vector valued semigroups,* Semigroup Forum **26** (1983), 65-74.

[2] G. Čupona, N. Celakoski, S. Markovski and D. Dimovski: *Vector valued groupoids, semigroups and groups,* in "Vector Valued Semigroups and Groups", Skopje (1988), Mac. Acad. Sci. Arts, 1-78.

[3] G. Čupona and D. Dimovski: *On a class of vector valued groups,* in "Algebra and logic", Zagreb (1984), 29-38.

[4] G. Čupona, D. Dimovski and A.Samardžiski: *Fully commutative vector valued groups,* Contributions - Sect. Math. Techn. Sci., MANU, **VIII 2** (1987), 5-17.

[5] G. Čupona, A. Samardžiski and N. Celakoski: *Fully commutative vector valued groupoids,* in "Algebra and Logic" Sarajevo 1987, Novi Sad (1989), 29-42.

[6] D. Dimovski: *Free vector valued semigroups,* in "Algebra and Logic" Cetinje 1986, Novi Sad (1987), 55-62.

[7] D. Dimovski: *Examples of* $(2m+s,m)$*-groups,* Contributions - Sect. Math. Techn. Sci., MANU, **VII 2** (1985), 105-114.

[8] D. Dimovski: *Free* $(n+1,n)$*-groups,* in "Vector Valued Semigroups and Groups", Skopje (1988), Mac. Acad. Sci. Arts, 103-122.

[9] D. Dimovski, B. Janeva and S.Ilić: *Free* (n,m)*-groups,* Communication in Algebra **19** (1991), 965-978.

[10] D. Dimovski and K. Trenčevski, *One-dimensional* $(4,2)$*-Lie groups,* in "Vector Valued Semigroups and Groups" Skopje (1988), Mac. Acad. Sci. Arts, 91-102.

[11] D. Dimovski and K. Trenčevski: *Nonexistence of continuous* $(4,3)$*-groups on R,* in "Algebra and Logic" Sarajevo 1987, Novi Sad (1989), 49-57.

[12] K. Trenčevski: *Some examples of fully commutative vector valued groups,* Contributions - Sect. Math. Techn. Sci., MANU, **IX 1-2** (1988), 27-37.

[13] K. Trenčevski: *A note on non-existence for some classes of continuous* $(3,2)$ *groups,* in "Algebra and Logic" Cetinje 1986, Novi Sad (1987), 205-208.

[14] K. Trenčevski and D. Dimovski: *"Complex Commutative Vector Valued Groups",* Mac. Acad. Sci. Arts, Skopje, 1992.

[15] K. Trenčevski and D. Dimovski: *On the affine and projective commutative* $(m+k,m)$*-groups,* Journal of Algebra **240** (2001), 338-365.

[16] B.Trpenovski and G.Čupona: $[m,n]$*-groupoids,* Bulletin of the the Assoc. of Math. and Phys. of Macedonia, **21** (1970), 19-29 (in Macedonian).

[17] J. Ušan: *Note on (n,m)-groups,* Math. Moravica **3** (1999), 27-139.

ЕДИНИЦИ ВО ВЕКТОРСКО ВРЕДНОСНИ ГРУПИ

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Резиме

Во овој чланок се разгледуваат единици на $(m+k,m)$-групи и $com(m+k,m)$-групи. Докажани се некои нивни својства. И во двата случаја изучувани се исто така и единици од облик $(e,e,\cdots,e)$. Во последниот параграф се дадени примери на $com(3,2)$-групи со единици.

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