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CHARACTERIZATION OF $(m+k,m)-$ RECTANGULAR BAND WHEN $k<m$

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Abstract. An $(m+k,m)-$ semigroup $(Q;[\ ])$ which is a direct product of a left-zero $(m+k,m)-$ semigroup and of a right-zero $(m+k,m)-$ semigroup is called an $(m+k,m)-$ rectangular band. In this paper we give a characterization of an $(m+k,m)-$ rectangular band, when $k<m$.

1. Introduction

Let Q be a nonempty set, and $k\geq 1$. The pair $(Q;[\ ])$ is called an $(m+k,m)-$ semigroup if $[\ ]:Q^{m+k}\to Q^m$ is a map satisfying the following condition:

$$[x_1^i[x_{i+1}^{i+m+k}]x_{i+m+k+1}^{m+2k}]=[[x_1^{m+k}]x_{m+k+1}^{m+2k}],\ \text{for each } 1\leq i\leq k.$$

Let $\boldsymbol{A}=(A;[\ ])$ be an $(m+k,m)$-groupoid, where $[\ ]$ is an $(m+k,m)-$ operation defined by $[x_1^{m+k}]=x_1^m$. Then $\boldsymbol{A}$ is an $(m+k,m)-$ semigroup and it is called a left-zero $(m+k,m)-$ semigroup. Dually, a right-zero $(m+k,m)-$ semigroup $(B;[\ ])$ is defined by the operation $[x_1^{m+k}]=x_{k+1}^{m+k}$.

The pair $(A\times B;[\ ])$, where $[\ ]$ is an $(m+k,m)-$ operation on $A\times B$ defined by

$$[x_1^{m+k}]=y_1^m \Leftrightarrow (x_i=(a_i,b_i), y_j=(a_j,b_{j+k}), i\in \lceil_{m+k},\ j\in\lceil_m)$$

is an $(m+k,m)-$ semigroup and it is a direct product of a left-zero and a right-zero $(m+k,m)-$ semigroup on A and B, respectively. Such an $(m+k,m)-$ semigroup is called $(m+k,m)-$ rectangular band.

2. Characterization of $(m+k,m)-$ rectangular band when $k<m$

A characterization of $(2k,k)$-rectangular bands is given in [3], i.e. the case when $k=m$. Here we will give a characterization of $(m+k,m)-$ rectangular band when $k<m$, and we assume bellow that $k<m$. First we state the following lemma:

Lemma 1. Let $\mathbf{Q}=(Q;[\ ])$ be an $(m+k,m)-$ semigroup, and $\mathbf{Q}$ satisfy the following equalities:

(a) $[x_1^{m+2k}]_i=[x_1^i x_{i+k+1}^{m+2k}]_i$, $i\in\lceil_m$;

(b) $[x_1^{m+k}]_i=[y_1^{j-1}x_i y_{j+1}^{j+k-1}x_{i+k}y_{j+k+1}^{m+k}]_j$, $i,j\in\lceil_m$.

Then $[x_1^{m+sk}]_1=[x_1x_{(s-1)k+2}^{m+sk}]_1$, $s\geq 2$.

Proof: The proof will be given by induction on s. Since $\mathbf{Q}$ satisfies (a), it follows that the case $s=2$ is true. Let the statement hold for $s-1$ and a be a fixed element of Q. Then:

$$[x_1^{m+sk}]_1 = [x_1^k [x_{k+1}^{m+sk}]]_1 \overset{(b)}{=} [x_1^k [x_{k+1}^{m+sk}]_1 \overset{m-1}{a}]_1 \overset{IH}{=} [x_1^k [x_{k+1} x_{(s-1)k+2}^{m+sk}]_1 \overset{m-1}{a}]_1 \overset{(b)}{=}$$

$$= [x_1^k [x_{k+1} x_{(s-1)k+2}^{m+sk}]_1 [x_{k+1} x_{(s-1)k+2}^{m+sk}]_2 \ldots [x_{k+1} x_{(s-1)k+2}^{m+sk}]_m]_1 = [x_1^k x_{k+1} x_{(s-1)k+2}^{m+sk}]_1 \overset{(a)}{=}$$

$$= [x_1 x_{(s-1)k+2}^{m+sk}]_1 . \ \blacksquare$$

Proposition 2. Let $\mathbf{Q} = (Q;[\ ])$ be an $(m+k,m)-$semigroup. $\mathbf{Q}$ is an $(m+k,m)-$rectangular band if and only if the conditions (a), (b) of Lemma 1. and (c) $[\overset{m+k}{x}] = \overset{m}{x}$ are satisfied in $\mathbf{Q}$.

Proof: Suppose that the $(m+k,m)-$semigroup $\mathbf{Q} = (Q;[\ ])$, satisfies (a), (b), (c), l is the least positive integer such that $m \le lk$ and $t = lk - m$, and a is a fixed element of Q.

(A) Denote by L the subset of Q $L = \{[x \overset{m+k-1}{a}]_1 \mid x \in Q\}$. Let $[x_i \overset{m+k-1}{a}]_1 \in L$, $i \in \lceil_{m+k}$. Then:

$$[[x_1 \overset{m+k-1}{a}]_1 \ldots [x_{m+k} \overset{m+k-1}{a}]_1]_i \overset{(b)}{=} [[x_i \overset{m+k-1}{a}]_1 \overset{k-1}{a} [x_{i+k} \overset{m+k-1}{a}]_1 \overset{m-1}{a}]_1 \overset{(b)}{=}$$

$$[[x_i \overset{m+k-1}{a}]_1 \overset{k-1}{a} [x_{i+k} \overset{m+k-1}{a}]_1 [x_{i+k} \overset{m+k-1}{a}]_2 \ldots [x_{i+k} \overset{m+k-1}{a}]_m]_1 = [[x_i \overset{m+k-1}{a}]_1 \overset{k-1}{a} x_{i+k} \overset{m+k-1}{a}]_1 \overset{(a)}{=}$$

$$= [[x_i \overset{m+k-1}{a}]_1 \overset{m+k-1}{a}]_1 \overset{L1}{=} [[x_i \overset{m+k-1}{a}]_1 \ldots [x_i \overset{m+k-1}{a}]_m \overset{t+1}{a} \overset{m+k-1}{a}]_1 = [x_i \overset{m+k-1}{a} \overset{t+1}{a} \overset{m+k-1}{a}]_1 \overset{L1}{=}$$

$$[x_i \overset{m+k-1}{a}]_1 .$$

So, $(L;[\ ])$ is a left-zero $(m+k,m)-$semigroup.

(B) Let $D = \{[\overset{m+k-1}{a} x]_m \mid x \in Q\}$ and $[\overset{m+k-1}{a} x_i]_m \in D$, $i \in \lceil_{m+k}$. Then:

$$[[\overset{m+k-1}{a} x_1]_m \ldots [\overset{m+k-1}{a} x_{m+k}]_m]_i \overset{(b)}{=} [[\overset{m+k-1}{a} x_i]_m \overset{k-1}{a} [\overset{m+k-1}{a} x_{i+k}]_m \overset{m-1}{a}]_1 \overset{(b)}{=}$$

$$= [[\overset{k}{a} x_i \overset{m-1}{a}]_1 \overset{k-1}{a} [\overset{k}{a} x_{i+k} \overset{m-1}{a}]_1 \overset{m-1}{a}]_1 \overset{(b)}{=} [[\overset{k}{a} x_i \overset{m-1}{a}]_1 \overset{k-1}{a} [\overset{k}{a} x_{i+k} \overset{m-1}{a}]_1 \ldots [\overset{k}{a} x_{i+k} \overset{m-1}{a}]_m]_1 =$$

$$= [[\overset{k}{a} x_i \overset{m-1}{a}]_1 \overset{k-1}{a} \overset{k}{a} x_{i+k} \overset{m-1}{a}]_1 \overset{(a)}{=} [[\overset{k}{a} x_i \overset{m-1}{a}]_1 \overset{k-1}{a} x_{i+k} \overset{m-1}{a}]_1 \overset{L1}{=}$$

$$= [[\overset{k}{a} x_i \overset{m-1}{a}]_1 \ldots [\overset{k}{a} x_i \overset{m-1}{a}]_m \overset{t+1}{a} \overset{k-1}{a} x_{i+k} \overset{m-1}{a}]_1 = [\overset{k}{a} x_i \overset{m-1}{a} \overset{t+1}{a} \overset{k-1}{a} x_{i+k} \overset{m-1}{a}]_1 \overset{L1}{=} [a \overset{k-1}{a} x_{i+k} \overset{m-1}{a}]_1 \overset{(b)}{=}$$

$$= [\overset{m+k-1}{a} x_{i+k}]_m .$$

So, $(D;[\ ])$ is a right-zero $(m+k,m)-$semigroup.

(C) We define a map $\varphi : L \times D \to Q$ with:

$$(\forall([x \overset{m+k-1}{a}]_1, [\overset{m+k-1}{a} y]_m) \in L \times D)\ \varphi([x \overset{m+k-1}{a}]_1, [\overset{m+k-1}{a} y]_m) = [x \overset{k-1}{a} y \overset{m-1}{a}]_1 .$$

(C1) We will prove that φ is a well-defined map.

Let $[x \overset{m+k-1}{a}]_1 = [u \overset{m+k-1}{a}]_1$, $[\overset{m+k-1}{a} y]_m = [\overset{m+k-1}{a} v]_m$. Then:

(C1.1) $$[[x \overset{m+k-1}{a}]_1 \overset{m+k-1}{x}]_1 = [[u \overset{m+k-1}{a}]_1 \overset{m+k-1}{x}]_1$$

$$[[x \overset{m+k-1}{a}]_1 \ldots [x \overset{m+k-1}{a}]_m \overset{t+1}{a} \overset{m+k-1}{x}]_1 = [[u \overset{m+k-1}{a}]_1 \ldots [u \overset{m+k-1}{a}]_k \overset{t+1}{a} \overset{m+k-1}{x}]_1$$

$$[x \overset{m+k-1}{a} \overset{t+1}{a} \overset{m+k-1}{x}]_1 = [u \overset{m+k-1}{a} \overset{t+1}{a} \overset{m+k-1}{x}]_1$$

$$[x \overset{m+k-1}{x}]_1 = [u \overset{m+k-1}{x}]_1$$

So, $x = [u \overset{m+k-1}{x}]_1$.

(C1.2) $$[\overset{m+k-1}{y} [\overset{m+k-1}{a} y]_m]_m = [\overset{m+k-1}{y} [\overset{m+k-1}{a} v]_m]_m$$

$$[\overset{k}{y}[\overset{m+k-1}{a} y]_m \overset{m-1}{y}]_1 = [\overset{k}{y}[\overset{m+k-1}{a} v]_m \overset{m-1}{y}]_1$$

$$[\overset{k}{y}[\overset{k}{a} y \overset{m-1}{a}]_1 \overset{m-1}{y}]_1 = [\overset{k}{y}[\overset{k}{a} v \overset{m-1}{a}]_1 \overset{m-1}{y}]_1$$

$$[\overset{k}{y}[\overset{k}{a} y \overset{m-1}{a}]_1 \ldots [\overset{k}{a} y \overset{m-1}{a}]_m]_1 = [\overset{k}{y}[\overset{k}{a} v \overset{m-1}{a}]_1 \ldots [\overset{k}{a} v \overset{m-1}{a}]_m]_1$$

$$[\overset{k}{y} \overset{k}{a} y \overset{m-1}{a}]_1 = [\overset{k}{y} \overset{k}{a} v \overset{m-1}{a}]_1$$

$$[\overset{k-1}{y} a y \overset{m-1}{a}]_1 = [\overset{k-1}{y} a v \overset{m-1}{a}]_1$$

$$[\overset{k-1}{y} y y \overset{m-1}{y}]_1 = [\overset{k-1}{y} y v \overset{m-1}{y}]_1 .$$

So, $y = [\overset{k}{y} v \overset{m-1}{y}]_1$.

Then:

$[x \overset{k-1}{a} y \overset{m-1}{a}]_1 = [[u \overset{m+k-1}{x}]_1 \overset{k-1}{a} [\overset{k}{y} v \overset{m-1}{y}]_1 \overset{m-1}{a}]_1 \overset{(b)}{=} [[u \overset{m+k-1}{x}]_1 \overset{k-1}{a} [\overset{k}{y} v \overset{m-1}{y}]_1 \ldots [\overset{k}{y} v \overset{m-1}{y}]_m]_1 =$

$= [[u \overset{m+k-1}{x}]_1 \overset{k-1}{a} \overset{k}{y} v \overset{m-1}{y}]_1 \overset{(a)}{=} [[u \overset{m+k-1}{x}]_1 \overset{k-1}{y} v \overset{m-1}{y}]_1 \overset{L1}{=} [[u \overset{m+k-1}{x}]_1 \ldots [u \overset{m+k-1}{x}]_m \overset{t+1}{a} \overset{k-1}{y} v \overset{m-1}{y}]_1 =$

$= [u \overset{m+k-1}{x} \overset{t+1}{a} \overset{k-1}{y} v \overset{m-1}{y}]_1 \overset{L1}{=} [u \overset{k-1}{y} v \overset{m-1}{y}]_1 .$

(C2) We will prove that φ is an injection.

Let $\varphi([x \overset{m+k-1}{a}]_1, [\overset{m+k-1}{a} y]_m) = \varphi([u \overset{m+k-1}{a}]_1, [\overset{m+k-1}{a} v]_m)$, i.e. $[x \overset{k-1}{a} y \overset{m-1}{a}]_1 = [u \overset{k-1}{a} v \overset{m-1}{a}]_1$.

Then:

$$[[x \overset{k-1}{a} y \overset{m-1}{a}]_1 \overset{m+k-1}{x}]_1 = [[u \overset{k-1}{a} v \overset{m-1}{a}]_1 \overset{m+k-1}{x}]_1$$

$$[[x \overset{k-1}{a} y \overset{m-1}{a}]_1 \ldots [x \overset{k-1}{a} y \overset{m-1}{a}]_m \overset{t+1}{a} \overset{m+k-1}{x}]_1 = [[u \overset{k-1}{a} v \overset{m-1}{a}]_1 \ldots [u \overset{k-1}{a} v \overset{m-1}{a}]_m \overset{t+1}{a} \overset{m+k-1}{x}]_1$$

$$[x \overset{k-1}{a} y \overset{m-1}{a} \overset{t+1}{a} \overset{m+k-1}{x}]_1 = [u \overset{k-1}{a} v \overset{m-1}{a} \overset{t+1}{a} \overset{m+k-1}{x}]_1$$

$$[x \overset{m+k-1}{x}]_1 = [u \overset{m+k-1}{x}]_1 \text{ i.e.}$$

$$x = [u \overset{m+k-1}{x}]_1 .$$

So, $[x \overset{m+k-1}{a}]_1 = [[u \overset{m+k-1}{x}]_1 \overset{m+k-1}{a}]_1 \overset{L1}{=} [[u \overset{m+k-1}{x}]_1 \ldots [u \overset{m+k-1}{x}]_m \overset{t+1}{a} \overset{m+k-1}{a}]_1 =$

$= [u \overset{m+k-1}{x} \overset{t+1}{a} \overset{m+k-1}{a}]_1 \overset{L1}{=} [u \overset{m+k-1}{a}]_1 .$

Similarly:

$$[\overset{m+k-1}{y}[x\overset{k-1}{a}y\overset{m-1}{a}]_1]_m = [\overset{m+k-1}{y}[u\overset{k-1}{a}v\overset{m-1}{a}]_1]_m$$

$$[\overset{k}{y}[x\overset{k-1}{a}y\overset{m-1}{a}]_1\overset{m-1}{a}]_1 = [\overset{k}{y}[u\overset{k-1}{a}v\overset{m-1}{a}]_1\overset{m-1}{a}]_1$$

$$[\overset{k}{y}[x\overset{k-1}{a}y\overset{m-1}{a}]_1\ldots[x\overset{k-1}{a}y\overset{m-1}{a}]_m]_1 = [\overset{k}{y}[u\overset{k-1}{a}v\overset{m-1}{a}]_1\ldots[u\overset{k-1}{a}v\overset{m-1}{a}]_m]_1$$

$$[\overset{k}{y}x\overset{k-1}{a}y\overset{m-1}{a}]_1 = [\overset{k}{y}u\overset{k-1}{a}v\overset{m-1}{a}]_1$$

$$[y\overset{k-1}{a}y\overset{m-1}{a}]_1 = [y\overset{k-1}{a}v\overset{m-1}{a}]_1$$

$$[y\overset{k-1}{y}y\overset{m-1}{y}]_1 = [y\overset{k-1}{y}v\overset{m-1}{y}]_1 .$$

So, $y=[\overset{k}{y}v\overset{m-1}{y}]_1$. Then $[\overset{m+k-1}{a}y]_m = [\overset{m+k-1}{a}[\overset{k}{y}v\overset{m-1}{y}]_1]_m \overset{(b)}{=} [\overset{k}{a}[\overset{k}{y}v\overset{m-1}{y}]_1\overset{m-1}{a}]_1 \overset{(b)}{=}$
$= [\overset{k}{a}[\overset{k}{y}v\overset{m-1}{y}]_1\ldots[\overset{k}{y}v\overset{m-1}{y}]_m]_1 = [\overset{k}{a}\overset{k}{y}v\overset{m-1}{y}]_1 \overset{(a)}{=} [a\overset{k-1}{y}v\overset{m-1}{y}]_1 \overset{(b)}{=} [\overset{m+k-1}{a}v]_m$.

So $([x\overset{m+k-1}{a}]_1,[\overset{m+k-1}{a}y]_m) = ([u\overset{m+k-1}{a}]_1,[\overset{m+k-1}{a}v]_m)$, i.e. φ is an injection.

(C3) We will prove that φ is a surjection.

Let $x\in Q$. Then $[x\overset{m+k-1}{a}]_1\in L$, $[\overset{m+k-1}{a}x]_m\in D$ and $\varphi([x\overset{m+k-1}{a}]_1,[\overset{m+k-1}{a}x]_m)=$
$=[x\overset{k-1}{a}x\overset{m-1}{a}]_1 \overset{(b)}{=} [\overset{m+k}{x}]_1 \overset{(c)}{=} x$. So, φ is a surjection.

(C4) We will prove that φ is an $(m+k,m)-$homomorphism.

Let $\alpha_i = ([x_i\overset{m+k-1}{a}]_1,[\overset{m+k-1}{a}y_i]_m)\in L\times D$, $i\in \lceil_{m+k}$. Then:

$[\alpha_1^{m+k}]_i = [([x_1\overset{m+k-1}{a}]_1,[\overset{m+k-1}{a}y_1]_m)\ldots([x_{m+k}\overset{m+k-1}{a}]_1,[\overset{m+k-1}{a}y_{m+k}]_m)]_i =$
$=([[x_1\overset{m+k-1}{a}]_1\ldots[x_{m+k}\overset{m+k-1}{a}]_1]_i,[[\overset{m+k-1}{a}y_1]_m\ldots[\overset{m+k-1}{a}y_{m+k}]_m]_i)$
$=([x_i\overset{m+k-1}{a}]_1,[\overset{m+k-1}{a}y_{i+k}]_m)$.

So, $\varphi([\alpha_1^{m+k}]_i)=\varphi([x_i\overset{m+k-1}{a}]_1,[\overset{m+k-1}{a}y_{i+k}]_m)=[x_i\overset{k-1}{a}y_{i+k}\overset{m-1}{a}]_1$.

We have $[\varphi(\alpha_1)\varphi(\alpha_2)\ldots\varphi(\alpha_{m+k})]_i = [[x_1\overset{k-1}{a}y_1\overset{m-1}{a}]_1\ldots[x_{m+k}\overset{k-1}{a}y_{m+k}\overset{m-1}{a}]_1]_i \overset{(b)}{=}$
$=[[x_i\overset{k-1}{a}y_i\overset{m-1}{a}]_1\overset{k-1}{a}[x_{i+k}\overset{k-1}{a}y_{i+k}\overset{m-1}{a}]_1\overset{m-1}{a}]_1 \overset{(b)}{=}$
$=[[x_i\overset{k-1}{a}y_i\overset{m-1}{a}]_1\overset{k-1}{a}[x_{i+k}\overset{k-1}{a}y_{i+k}\overset{m-1}{a}]_1\ldots[x_{i+k}\overset{k-1}{a}y_{i+k}\overset{m-1}{a}]_m]_1 =$
$=[[x_i\overset{k-1}{a}y_i\overset{m-1}{a}]_1\overset{k-1}{a}x_{i+k}\overset{k-1}{a}y_{i+k}\overset{m-1}{a}]_1 \overset{(a)}{=} [[x_i\overset{k-1}{a}y_i\overset{m-1}{a}]_1\overset{k-1}{a}y_{i+k}\overset{m-1}{a}]_1 \overset{L1}{=}$
$=[[x_i\overset{k-1}{a}y_i\overset{m-1}{a}]_1\ldots[x_i\overset{k-1}{a}y_i\overset{m-1}{a}]_m\overset{t+1}{a}\overset{k-1}{a}y_{i+k}\overset{m-1}{a}]_1 = [x_i\overset{k-1}{a}y_i\overset{m-1}{a}\overset{t+1}{a}\overset{k-1}{a}y_{i+k}\overset{m-1}{a}]_1 \overset{L1}{=}$
$=[x_i\overset{k-1}{a}y_{i+k}\overset{m-1}{a}]_1$.

So, $\varphi([\alpha_1^{m+k}]_i)=[\varphi(\alpha_1)\varphi(\alpha_2)\ldots\varphi(\alpha_{m+k})]_i$, i.e. φ is $(m+k,m)-$homomorphism.

Hence, $\mathbf{Q}$ is a direct product of a left-zero and a right-zero $(m+k,m)-$semigroup.

Conversely, let $\mathbf{Q}$ be a direct product of a left-zero and a right-zero $(m+k,m)-$semigroup.

(D) Let $(x_i, y_i) \in Q$, $i \in \lceil_{m+2k}$. Then:

$[(x_1,y_1)...(x_{m+2k},y_{m+2k})]_i =$

$[(x_1,y_{1+k})...(x_m,y_{m+k})(x_{m+k+1},y_{m+k+1})...(x_{m+2k},y_{m+2k})]_i =$

$=(x_i,y_{i+2k})$ and $[(x_1,y_1)...(x_i,y_i)(x_{i+k+1},y_{i+k+1})...(x_{m+2k},y_{m+2k})]_i=(x_i,y_{i+2k})$.

Hence, $\mathbf{Q}$ satisfies (a).

(E) Let $(x_\alpha,y_\alpha),(a_\beta,b_\beta)\in Q$, $(x_i,y_i)=(a_j,b_j),(x_{i+k},y_{i+k})=(a_{j+k},b_{j+k})$, $\alpha,\beta \in \lceil_{m+k}$, $i,j \in \lceil_m$. Then:

$[(x_1,y_1)...(x_i,y_i)...(x_{i+k},y_{i+k})...(x_{m+k},y_{m+k})]_i=(x_i,y_{i+k})=$

$=[(a_1,b_1)...(a_{j-1},b_{j-1})(x_i,y_i)...(a_{j+k-1},b_{j+k-1})(x_{i+k},y_{i+k})...(a_{m+k},b_{m+k})]_j$.

Hence, $\mathbf{Q}$ satisfies (b).

(F) Let $(a,b)\in Q$. Then $[\overset{m+k}{(x,y)}]_i=(x,y)$ i.e. $[\overset{m+k}{(x,y)}]=\overset{m}{(x,y)}$. Hence, $\mathbf{Q}$ satisfies (c). ■

Proposition 3. Let $\mathbf{Q}=(Q;[\ ])$ be an $(m+k,m)-$semigroup, $k<m$. Then $\mathbf{Q}$ is a direct product of a left-zero and a right-zero $(m+k,m)-$semigroup if and only if there is a semigroup $(Q;*)$ which is a rectangular band, i.e. a direct product of a left-zero and a right-zero semigroup, such that $[x_1^{m+k}]_i = x_i * x_{i+k}$, $x_1^{m+k}\in Q^{m+k}$, $i\in \lceil_m$.

Proof: Suppose $\mathbf{Q}=(Q;[\ ])$ is an $(m+k,m)-$semigroup, direct product of a left-zero and a right-zero $(m+k,m)-$semigroup. According to Proposition 2. (a), (b) and (c) are satisfied in $\mathbf{Q}$.

For a fixed $a\in Q$, let $*$ be an operation defined on Q, by $x*y=[x\overset{k-1}{a}\ y\overset{m-1}{a}]_1$.

(A) Clearly $(Q;*)$ is groupoid.

(B) We will prove that $(Q;*)$ is a semigroup. Let $x,y,z\in Q$. Then:

$$(x*y)*z=[[x\overset{k-1}{a}\ y\overset{m-1}{a}]_1\overset{k-1}{a}\ z\overset{m-1}{a}]_1\overset{L1}{=}[[x\overset{k-1}{a}\ y\overset{m-1}{a}]_1...[x\overset{k-1}{a}\ y\overset{m-1}{a}]_m\overset{t+1}{a}\overset{k-1}{a}\ z\overset{m-1}{a}]_1=$$

$$=[x\overset{k-1}{a}\ y\overset{m-1}{a}\overset{t+1}{a}\overset{k-1}{a}\ z\overset{m-1}{a}]_1\overset{L1}{=}[x\overset{k-1}{a}\ z\overset{m-1}{a}]_1;$$

$$x*(y*z)=[x\overset{k-1}{a}[y\overset{k-1}{a}\ z\overset{m-1}{a}]_1\overset{m-1}{a}]_1\overset{(b)}{=}[x\overset{k-1}{a}[y\overset{k-1}{a}\ z\overset{m-1}{a}]_1...[y\overset{k-1}{a}\ z\overset{m-1}{a}]_m]_1=$$

$$=[x\overset{k-1}{a}\ y\overset{k-1}{a}\ z\overset{m-1}{a}]_1\overset{(a)}{=}[x\overset{k-1}{a}\ z\overset{m-1}{a}]_1.$$

Hence, $(x*y)*z=x*(y*z)$, i.e. $(Q;*)$ is a semigroup.

(C) We have $x*y*z=[x\overset{k-1}{a}\ z\overset{m-1}{a}]_1=x*z$ and $x*x=[x\overset{k-1}{a}\ x\overset{m-1}{a}]_1\overset{(b)}{=}$ $=[x\overset{k-1}{x}\ x\overset{m-1}{x}]_1\overset{(c)}{=}x$ in $(Q;*)$.

Hence, $(Q;*)$ is a semigroup in which $x*y*z = x*z$ and $x*x = x$, i.e. $(Q;*)$ is a rectangular band.

(D) Finally, we have $[x_1^{m+k}]_i \overset{(b)}{=} [x_i \overset{k-1}{a} x_{i+k} \overset{m-1}{a}]_1 = x_i * x_{i+k}$, for $x_1^{m+k} \in Q^{m+k}$, $i \in \lceil_m$. So, $(Q;*)$ is a semigroup, is a direct product of a left-zero and a right-zero semigroup and $[x_1^k y_1^k]_i = x_i * y_i$.

Conversely, let $(Q;[\ ])$ be an $(m+k,m)-$semigroup, and there is a semigroup $(Q;*)$ which is a rectangular band, such that $[x_1^{m+k}]_i = x_i * x_{i+k}$, $x_1^{m+k} \in Q^{m+k}$, $i \in \lceil_m$.

Then

(d) $x*y*z = x*z$ and

(e) $x*x = x$

hold in $(Q;*)$.

We will prove that the statements (*a*), (*b*) and (*c*) from Proposition 2. are true in $\mathbf{Q}$.

(E) Let $x_1^{m+2k} \in Q^{m+2k}$, $i \in \lceil_m$. Then:

$$[x_1^{m+2k}]_i = [x_1 * x_{1+k}, \dots, x_m * x_{m+k}, x_{m+k+1}, \dots, x_{m+2k}]_i = x_i * x_{i+k} * x_{i+2k} \overset{(d)}{=}$$
$$x_i * x_{i+2k} =$$
$$= [x_1^i x_{i+k+1}^{m+2k}]_i .$$

So, the statement (*a*) from Proposition 2. is true.

(F) Let $x_1^{m+k}, y_1^{m+k} \in Q^{m+k}$, $y_j = x_i$, $y_{j+k} = x_{i+k}$, $i, j \in \lceil_m$. Then: $[y_1^{j-1} x_i y_{j+1}^{j+k-1} x_{i+k} y_{j+k+1}^{m+k}]_j = x_i * x_{i+k} = [x_1^{m+k}]_i$. So, the statement (*b*) from Proposition 2. is true.

(G) Let $x \in Q$. Then: $[\overset{m+k}{x}]_i = x * x \overset{(e)}{=} x$, i.e. $[\overset{m+k}{x}] = \overset{m}{x}$. So, the statement (*c*) from Proposition 2. is true. ∎

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