FREE $(m+k,m)$ – RECTANGULAR BANDS WHEN $k<m$

Valentina Miovska, Donco Dimovski

Abstract: *A characterization of* $(m+k,m)$ – *rectangular bands when* $k<m$, *using the usual rectangular bands is given in [4]. This result is used to obtain a free* $(m+k,m)$ – *rectangular band when* $k<m$.

Keywords: *rectangular band,* $(m+k,m)$ – *rectangular band, free* $(m+k,m)$ – *rectangular band*

1. INTRODUCTION

First, we will introduce some notations which will be used further on:

1) The elements of Q^s, where Q^s denotes the s – th Cartesian power of Q, will be denoted by x_1^s.
2) The symbol x_i^j will denote the sequence $x_i x_{i+1} ... x_j$ when $i \le j$, and the empty sequence when $i > j$.
3) If $x_1 = x_2 = ... = x_s = x$, then x_1^s is denoted by the symbol $\overset{s}{x}$.
4) The set $\{1,2,...,s\}$ will be denoted by $\mathbf{N_s}$.

Let $Q \ne \varnothing$ and n,m be positive integers. If [ ] is a map from Q^n into Q^m, then [ ] is called an (n,m) – operation. A pair $(Q;[\ ])$ where [ ] is an (n,m) – operation is said to be an (n,m) – groupoid. Every (n,m) – operation on Q induces a sequence $[\ ]_1, [\ ]_2, ..., [\ ]_m$ of n – ary operations on the set Q, such that

$$((\forall i \in \mathbf{N_m})\ \lfloor x_1^n \rfloor_i = y_i) \Leftrightarrow \lfloor x_1^n \rfloor = y_1^m .$$

Let $m \ge 2, k \ge 1$. An $(m+k,m)$ – groupoid $(Q;[\ ])$ is called an $(m+k,m)$ – semigroup if for each $i \in \{0,1,2,...,k\}$

$$\lfloor x_1^i \lfloor x_{i+1}^{i+m+k} \rfloor x_{i+m+k+1}^{m+2k} \rfloor = \lfloor \lfloor x_1^{m+k} \rfloor x_{m+k+1}^{m+2k} \rfloor .$$

Let $(A;[\ ])$ be an $(m+k,m)$ – groupoid, where [ ] is an $(m+k,m)$ – operation defined by $[x_1^{m+k}] = x_1^m$. Then $(A;[\ ])$ is an $(m+k,m)$ – semigroup and it is called a left-zero $(m+k,m)$ – semigroup. Dually, a right-zero $(m+k,m)$ – semigroup $(B;[\ ])$ is defined by the operation $[x_1^{m+k}] = x_{k+1}^{m+k}$.

The pair $(A \times B;[\ ])$, where [ ] is an $(m+k,m)$ – operation on $A \times B$ defined by

$$[x_1^{m+k}] = y_1^m \ \Leftrightarrow\ (x_i = (a_i, b_i), y_j = (a_j, b_{j+k}), i \in \mathbf{N_{m+k}},\ j \in \mathbf{N_m})$$

is an $(m+k,m)$ – semigroup and it is a direct product of a left-zero and a right-zero $(m+k,m)$ – semigroup on A and B, respectively. Such an $(m+k,m)$ – semigroup is called $(m+k,m)$ – rectangular band.

The following propositions characterizes $(m+k,m)$ – rectangular bands when $k<m$.

Propositon 1.1 ([4, Proposition 2]) Let $\mathbf{Q}=(Q;[\ ])$ be an $(m+k,m)$ – semigroup, $k<m$. $\mathbf{Q}$ is an $(m+k,m)$ – rectangular band if and only if the conditions

(a) $\lfloor x_1^{m+2k} \rfloor_i = \lfloor x_1^i x_{i+k+1}^{m+2k} \rfloor_i$, $i \in \mathbf{N_m}$

(b) $\lfloor x_1^{m+k} \rfloor_i = \lfloor y_1^{j-1} x_i y_{j+1}^{j+k-1} x_{i+k} y_{j+k+1}^{m+k} \rfloor_j$, $i,j \in \mathbf{N_m}$

(c) $\left[\overset{m+k}{x}\right] = \overset{m}{x}$

are satisfied in $\mathbf{Q}$.

Propositon 1.2 ([4, Proposition 3]) Let $\mathbf{Q}=(Q;[\,])$ be an $(m+k,m)$ – semigroup, $k<m$. Then $\mathbf{Q}$ is a direct product of a left-zero and a right-zero $(m+k,m)$ – semigroup if and only if there is a rectangular band $(Q;*)$, such that $\lfloor x_1^{m+k} \rfloor_i = x_i * x_{i+k}$, $x_1^{m+k} \in Q^{m+k}$, $i \in \mathbf{N_m}$.

Propositon 1.2 gives a characterization of $(m+k,m)$ – rectangular bands using the usual rectangular bands. Rectangular band is a semigroup which is a direct product of a left-zero and a right-zero semigroup, or equivalent, rectangular band is a semigroup $(Q;*)$ that satisfies the following two identities $x*y*z=x*z$ and $x*x=x$, for each $x,y,z \in Q$.

This result of Propositon 1.2 is used to obtain a free $(m+k,m)$ – rectangular band when $k<m$.

2. Free $(m+k,m)$ – rectangular bands when $k<m$

Let $(Q;*)$ be a free rectangular band with a basis B. Then $Q=B\cup\{ab\,|\,a,b\in B, a\neq b\}$ and operation $*$ is defined by:

$$x * y = \begin{cases} a & \begin{array}{l} x = y = a \\ x = a,\ y = ca \\ x = ac,\ y = a \\ x = ac,\ y = da \end{array} \\ ab & \begin{array}{l} x = a,\ y = b \\ x = a,\ y = cb \\ x = ac,\ y = b \\ x = ac,\ y = db \end{array} ,\ a \neq b \end{cases}$$

Let $k < m$ and let [ ] be the $(m+k,m)-$operation on Q defined by

$$\lfloor x_1^{m+k} \rfloor_i = x_i * x_{i+k},\ x_1^{m+k} \in Q^{m+k},\ i \in \mathbf{N_m}.$$

Then

Proposition 2 $(Q;[\ ])$ is a free $(m+k,m)-$rectangular band when $k < m$ with a basis B.

Proof. Since $k < m$, let $k+t = m$, $t \geq 1$.

First, we will prove that $\lfloor\lfloor x_1^{m+k} \rfloor x_{m+k+1}^{m+2k} \rfloor_i = x_i * x_{i+2k}$.

a) Let $i \leq t$. Then $i+k \leq t+k = m$.

We have

$\lfloor\lfloor x_1^{m+k} \rfloor x_{m+k+1}^{m+2k} \rfloor_i =$

$= \lfloor x_1^{m+k} \rfloor_i * \lfloor x_1^{m+k} \rfloor_{i+k} =$

$= (x_i * x_{i+k}) * (x_{i+k} * x_{i+2k}) =$

$= x_i * x_{i+2k}$

b) Let $t < i \leq m$. Then $i = t + \lambda$, $1 \leq \lambda \leq k$ and $i+k = t+\lambda+k = m+\lambda$.

$\lfloor\lfloor x_1^{m+k} \rfloor x_{m+k+1}^{m+2k} \rfloor_i =$

$= \lfloor x_1^{m+k} \rfloor_i * x_{m+k+\lambda} =$

$= (x_i * x_{i+k}) * x_{m+k+\lambda} =$

$= x_i * x_{m+k+\lambda} =$

$= x_i * x_{i+2k}$.

Further on we will prove that $\lfloor x_1^j \lfloor x_{j+1}^{j+m+k} \rfloor x_{j+m+k+1}^{m+2k} \rfloor_i = x_i * x_{i+2k}$.

c) Let $i \leq j$. Then $i \leq j < j+t$ implies $i+k < j+t+k = j+m$. Moreover, $i+k > k \geq j$ i.e. $j < i+k < j+m$. Let $i+k = j+\lambda$.

We obtain

$\lfloor x_1^j \lfloor x_{j+1}^{j+m+k} \rfloor x_{j+m+k+1}^{m+2k} \rfloor_i =$

$= \lfloor x_1^i x_{i+1}^j \lfloor x_{j+1}^{j+m+k} \rfloor_1 \ldots \lfloor x_{j+1}^{j+m+k} \rfloor_\lambda \lfloor x_{j+1}^{j+m+k} \rfloor_{\lambda+1} \ldots \lfloor x_{j+1}^{j+m+k} \rfloor_m x_{j+m+k+1}^{m+2k} \rfloor_i =$

$= x_i * \lfloor x_{j+1}^{j+m+k} \rfloor_\lambda =$

$= x_i * (x_{j+\lambda} * x_{j+\lambda+k}) =$

$= x_i * x_{j+\lambda+k} =$

$= x_i * x_{i+2k}$.

d) Let $j < i$.

d1) Let $j+1 \le i \le j+t$ i.e. $i = j+\lambda$, $1 \le \lambda \le t$.

Then $i+k = j+\lambda+k \le j+t+k = j+m$.

$$\left[x_1^j \left[x_{j+1}^{j+m+k}\right] x_{j+m+k+1}^{m+2k}\right]_i =$$
$$= \left[x_{j+1}^{j+m+k}\right]_\lambda * \left[x_{j+1}^{j+m+k}\right]_{\lambda+k} =$$
$$= \left(x_{j+\lambda} * x_{j+\lambda+k}\right) * \left(x_{j+\lambda+k} * x_{j+\lambda+k+k}\right) =$$
$$= x_{j+\lambda} * x_{j+\lambda+k+k} =$$
$$= x_i * x_{i+2k}.$$

d2) Let $j+t < i$ i.e. $i = j+t+\lambda$, $1 \le \lambda \le k-j$. Then $j+t+k < i+k$ i.e. $j+m < i+k$.

$$\left[x_1^j \left[x_{j+1}^{j+m+k}\right] x_{j+m+k+1}^{m+2k}\right]_i =$$
$$= \left[x_{j+1}^{j+m+k}\right]_{t+\lambda} * x_{j+m+k+\lambda} =$$
$$= \left(x_{j+t+\lambda} * x_{j+t+\lambda+k}\right) * x_{j+k+t+k+\lambda} =$$
$$= x_{j+t+\lambda} * x_{j+k+t+k+\lambda} =$$
$$= x_i * x_{i+2k}.$$

Then $\left[\left[x_1^{m+k}\right] x_{m+k+1}^{m+2k}\right]_i = \left[x_1^j \left[x_{j+1}^{j+m+k}\right] x_{j+m+k+1}^{m+2k}\right]_i$, for any $i \in \mathbf{N_m}$, $0 \le j \le k$. So $(Q;[\ ])$ is an $(m+k, m)$ – semigroup, when $k < m$.

According to Proposition 1.2 $(Q;[\ ])$ is an $(m+k, m)$ – rectangular band, when $k < m$.

It is clear that $B \subseteq Q$. Let $u \in Q$ and let $[B]$ be an $(m+k, m)$ – subsemigroup of $(Q;[\ ])$ generated by B. Then, for $c \in B$ we have $u = ab = a * b = \left[\overset{i-1}{c}\, a\, \overset{k-1}{c}\, b\, \overset{m-i}{c}\right]_i \in [B]$, i.e. $Q \subseteq [B]$. So, $(Q;[\ ])$ is generated by B.

Let $(S;[\ ]')$ be an $(m+k, m)$ – rectangular band when $k < m$ and let $f : B \to S$ be a map. By Proposition 1.2, there is a rectangular band $(S;\circ)$ such that $\left[x_1^{m+k}\right]_i' = x_i \circ x_{i+k}$, $x_1^{m+k} \in S^{m+k}$, $i \in \mathbf{N_m}$. Since $(Q;*)$ is a free rectangular band with a basis B, there is a homomorphism $g : Q \to S$, such that $g(b) = f(b), b \in B$.

Let $x_1^{m+k} \in Q^{m+k}$. Then

$$g\left(\left[x_1^{m+k}\right]_i\right) = g\left(x_i * x_{i+k}\right) = g(x_i) \circ g(x_{i+k}) = \left[g(x_1) g(x_2) \ldots g(x_{m+k})\right]_i'.$$

So, g as an extension of the map f, is $(m+k, m)$ – homomorphism from $(Q;[\ ])$ into $(S;[\ ]')$. Hence $(Q;[\ ])$ is a free $(m+k, m)$ – rectangular band with a basis B when $k < m$. ■

REFERENCES

[1] Čupona G. (1983) Vector valued semigroups, Semigroup forum, Vol 26, 65-74.
[2] Čupona G., Celakoski N., Markovski S., Dimovski D. (1988) Vector valued groupoids, semigroups and groups, "Vector valued semigroups and groups", Maced. Acad. of Sci. and Arts, 1-79.
[3] Dimovski D., Miovska V. (2004) Characterization of (2*k*, *k*)– rectangular band, Matematički Bilten, 28(LIV), 125-132.
[4] Dimovski D., Miovska V. Characterization of $(m+k,m)$ – rectangular band when $k<m$, submitted in Proceeding of 3th Congress of Mathematicians of Macedonia, Struga, 29.09-02.10. 2005.