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# FRAGMENTABILITY OF FUNCTION SPACES $C_p(T)$ FOR PSEUDOCOMPACT SPACES $T$

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**Abstract.** For a compact space $T$ it is known that the space $C_p(T)$ (of all continuous functions in T, endowed with the pointwise convergence topology $p$) is fragmentable by a metric that majorizes $p$ if and only if it is fragmentable by another metric which majorizes the sup-norm topology in $C(T)$. We show that this fact remains valid for pseudocompact spaces $T$. For pseudocompact and for strongly pseudocompact spaces $T$ we give characterizations of fragmentability of $C_p(T)$ by means of a topological game which is a modification of a game used earlier for characterization of fragmentability. The results are based on a recent generalization of the theorem of Eberlein.

## 1. Preliminaries

A metric $d(.,.)$ defined in a topological space $X$ is said to fragment $X$, if for every $\varepsilon > 0$ and every non-empty subset $A \subset X$ there exists an open subset $U \subset X$ such that the set $A \cap U$ is not empty and its $d$-diameter is smaller than $\varepsilon$. i.e. every non-empty set $A \subset X$ contains relatively open subsets of arbitrarily small diameters. The space X is said to be fragmentable if there exists a metric that fragments it. Fragmentability was introduced by Jayne and Rogers (see [7]) and studied by many authors. It proved to be a convenient tool in the study of Banach spaces, differentiability of convex functions as well as in many topological contexts (see Jayne, Namioka and Rogers [8]–[12], Ribarska [16]–[18], Namioka [15] and Kenderov, Moors [13], [14]). Of a particular interest is the case when the open subsets of $X$ are open in the metric topology generated by the metric $d$. In such a case it is said that *$d$ majorizes the topology of $X$*.

For a compact space $T$ it has been shown (see [14] and [13]) that the function space $C_p(T)$, where $p$ stands for the pointwise convergence topology, is fragmented by a metric $d$ majorizing $p$ if, and only if, there exists another metric $d$ which fragments $C_p(T)$ and majorizes the uniform convergence topology (the one generated by the "sup-norm" in $C(T)$). The first goal of this paper is to show that this result remains valid for pseudocompact spaces $T$ as well. The second goal is to give one more game characterization of fragmentability of $C_p(T)$ (by a metric majorizing

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$p$) for the cases when $T$ is a pseudocompact or a strongly pseudocompact space.

Recall that a subset $A$ of a completely regular space $X$ is said to be bounded in $X$ if every continuous real-valued function defined on $X$ is bounded on $A$. If a completely regular space $X$ is bounded in itself, then it is called pseudocompact (Engelking [5], Theorem 3.10.22). Every countably compact space $T$ is pseudocompact. There are however many pseudocompact spaces which are not countably compact.

**Definition 1.1** ([3] Definition 1.2). *A subset $A$ of a topological space $X$ is called strongly bounded in $X$, if it contains a dense subset $D$ with the property that for every sequence $\{x_i\}_{i\geq 1}$ in $D$ there exists a subsequence which is bounded in some separable subspace $S$ of $X$. A space $X$ which is strongly bounded in itself is called strongly pseudocompact.*

Every strongly pseudocompact space $T$ is pseudocompact. There are however pseudocompact spaces which are not strongly pseudocompact. Shakhmatov [19] constructed a pseudocompact space $T$ such that the closed unit ball $B = \{x \in C(T) : \|x\| = \max_{t\in T} |x(t)| \leq 1\}$ is pseudocompact but not a compact subset of $C_p(T)$. As it follows from Theorem 1.2 below, neither of the pseudocompact spaces $T$ and $B$ (from the example of Shakhmatov) is strongly pseudocompact. This example outlines some limits for the possible generalizations of the theorem of Eberlein. Recall that, for a compact space $T$, Eberlein [4] has shown that the closure of every countably compact subset of $C_p(T)$ is compact. Grothendieck [6] proved that this result remains valid for countably compact spaces $T$. Another generalization was obtained by Asanov and Veličko [1] who have shown that, if $A$ is bounded in $C_p(T)$ and $T$ is countably compact, then $\overline{A}$ is a compact subset of $C_p(T)$.

As shown in [3] the notions "bounded" and "strongly bounded" provide a convenient framework for further generalizations of Eberlein Theorem.

**Theorem 1.1** ([3] Theorem 4.1 and Theorem 4.2). *Let $T$ be a completely regular pseudocompact (strongly pseudocompact) space and let $A$ be a nonempty set which is strongly bounded (bounded) in $C_p(T)$. Then*

*(i) $\overline{A}$ is a non-empty compact subset of $C_p(T)$;*

*(ii) Every sequence $\{f_i\}_{i\geq 1}$ of functions $f_i \in A$, $i \geq 1$, has a subsequence converging to some $f_0$ in $C_p(T)$. If, in addition, the sequence $\{f_i\}_{i\geq 1}$ is contained in some ball in $C(T)$, then for every $\epsilon > 0$ there exist an integer $k > 0$ and nonnegative numbers $\lambda_i$, $1 \leq i \leq k$, such that $\sum_{i=1}^{k} \lambda_i = 1$ and $|f_0(t) - \sum_{i=1}^{k} \lambda_i f_i(t)| \leq \epsilon$ for every $t \in T$.*

We will need one more statement which is also a generalization of the Eberlein theorem:

**Theorem 1.2** ([3], Theorem 3.2). *The conclusions (i) and (ii) of the above Theorem 1.1 remain valid, if $T$ is a pseudocompact space and $A$ is a set bounded in a separable subspace $S$ of the space $C_p(T)$.*

As in [14] and [13] our main tool for studying fragmentability is the following topological game.

Two players, $\Sigma$ and $\Omega$, play a game by selecting alternatively non-empty subsets of $X$. The player $\Sigma$ begins the game by selecting some non-empty subset $A_1$ of $X$. In turn, player $\Omega$ selects some non-empty relatively open subset $B_1$ of $A_1$. Then $\Sigma$ selects a non-empty subset $A_2 \subseteq B_1$ and, again, $\Omega$ choses a non-empty relatively open subset $B_2 \subseteq A_2$. Proceeding this way, the two players generate a nested sequence $\{(A_i, B_i)\}_{i1}$ of sets which we call a *play*. The player is said to have won the play $\{(A_i, B_i)\}_{i1}$ if either the intersection $\cap_{i\geq 1} A_i = \cap_{i\geq 1} B_i$ is empty or consists of just one point. Otherwise player $\Sigma$ is declared to be the winner of this play.

This game will be referred to as the *Fragmenting Game in* X and will be denoted by $G(X)$.

By a strategy $\omega$ for player $\Omega$ we mean "a rule" that specifies each move of this player in "every possible situation". The strategy $\omega$ is called winning, if every play generated by applying the strategy $\omega$ is won by the player $\Omega$. Similarly, one defines the notions "strategy" and "winning strategy" for the player $\Sigma$.

**Theorem 1.3** ([14])**.** *The topological space $X$ is fragmentable if, and only if, the player $\Omega$ has a winning strategy in the game $G(X)$.*

**Theorem 1.4** ([13])**.** *The topological space $(X, \tau)$ is fragmentable by a metric that majorizes some topology $\tau'$ in $X$ if, and only if, there exists a winning strategy for $\Omega$ such that, for every play $\{A_i, B_i\}_{i\geq 1}$ generated by this strategy, the intersection $\cap_{i\geq 1} A_i = \cap_{i\geq 1} B_i$ is either empty or it consists of just one point $x_0$ and every $\tau'$-open set $U \ni x_0$ contains some $A_n$ (and, hence, all sets $A_i$ for which $i \geq n$). In particular, if the intersection $\cap_{i\geq 1} A_i = \cap_{i\geq 1} B_i$ is not empty (and consists of just one point $x_0$), then every sequence $\{x_i\}_{i\geq 1}$, $x_i \in A_i$, $i \geq 1$, $\tau'$-converges to $x_0$.*

## 2. Fragmentability of $C_p(T)$ for pseudocompact spaces $T$

For the formulation of our results we need a modification of the game $G$. The modified game will be denoted by $G'(X)$. In this game the player $\Sigma$ plays as in $G(X)$ and selects non-empty subsets of the space $X$. If $A_i$ is the $i$-th move of $\Sigma$, the player $\Omega$ answers as in $G$ by selecting a non-empty relatively open $B_i \subset A_i$ but selects also, in addition, a point $x_i \in A_i$.

**Definition 2.1.** *The player $\Omega$ is said to have won the play $\{(A_i, B_i, x_i)\}_{i\geq 1}$ in the game $G'(X)$ if either the intersection $\cap_{i\geq 1} A_i = \cap_{i\geq 1} B_i$ is empty or, if not empty, the sequence $\{x_i\}_{i\geq 1}$ contains a subsequence which is bounded in some separable subset of $X$.*

The notion of "strategy" and "winning strategy" are defined as above.

**Theorem 2.1.** *Let $T$ be a pseudocompact space. Then the following statements are equivalent:*

- *(i) The space $C_p(T)$ is fragmentable by a metric majorizing the norm topology of $C(T)$.*

*(ii)* *The space $C_p(T)$ is fragmentable by a metric majorizing the topology $p$.*

*(iii)* *The player $\Omega$ has a strategy $\omega$ in the game $G(C_p(T))$ which generates plays $\{(A_i, B_i)\}_{i\geq 1}$ such that either the intersection $\cap_{i\geq 1} A_i = \cap_{i\geq 1} B_i$ is empty or, otherwise, every sequence $\{g_i\}_{i\geq 1}$ with $g_i \in A_i$, $i \geq 1$, has a cluster point in $C_p(T)$.*

*(iv)* *The player $\Omega$ has a winning strategy $\omega'$ in the game $G'(C_p(T))$, i.e. $\Omega$ has a strategy $\omega'$ which generates plays $\{(A_i, B_i, g_i)\}_{i\geq 1}$ such that either the intersection $\cap_{i\geq 1} A_i = \cap_{i\geq 1} B_i$ is empty or, otherwise, the sequence $\{g_i\}_{i\geq 1}$ contains a subsequence which is bounded in some separable subset of $C_p(T)$.*

*Proof.* The implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are evident. In view of Theorem 1.4 the implication $(iv) \Rightarrow (i)$ follows from the next assertion. □

**Proposition 2.1.** *For any pseudocompact space $T$ the property $(iv)$ from Theorem 2.1 is equivalent to the following one:*

*(v)* *Player $\Omega$ has a strategy $\omega$ in the game $G(C_p(T))$ which generates plays $\{(A_i, B_i)\}_{i\geq 1}$ such that either the intersection $\cap_{i\geq 1} A_i = \cap_{i\geq 1} B_i$ is empty or, otherwise, the sup-norm diameters of the set $A_i$ tend to zero.*

*Proof.* Evidently, $(v) \Rightarrow (iv)$. It suffices (see Proposition 2.1 in [13]) to prove the implication $(iv) \Rightarrow (v)$ for the case when the games $G$ and $G'$ are played in the closed unit ball $B = \{f \in C(T) : \|f\| \leq 1\}$ endowed with the pointwise convergence topology $p$. We take an arbitrary strategy $\omega'$ for the game $G'((B, p))$ and construct a strategy $\omega$ for the game $G((B, p))$. Then we show that, if $\omega'$ satisfies $(iv)$ the constructed strategy $\omega$ satisfies $(v)$.

The construction of the strategy $\omega$ has its roots in the work of Christensen (see [2]) who proved that some continuous mappings into $C_p(T)$ are sup-norm continuous at many points. Here we follow [13] where the construction was adapted for the needs of fragmentability theory. The construction of the strategy $\omega$ uses induction. Let $A_1$ be an arbitrary first choice of player $\Sigma$ in the game $G((B, p))$. Using his/her strategy $\omega'$ the player $\Omega$ selects some $g_1 \in A_1$ and a relatively open subset $B'_1 \subset A_1$. Put $d_1 := \inf\{t > 0 : g_1 + t_B \supseteq B'_1\}$. If $d_1 = 0$, then $B'_1 = g_1$ and we take $\omega(A_1)$, the answer of $\Omega$, to be the relatively open subset $B_1 = g_1$. Note that, in this case, all subsequent moves $A_i$, $B_i$, $i \geq 2$, of the players are predetermined and trivial: $A_i = B_i = g_1$. Such plays are won by $\Omega$ in the sense of $(v)$. Therefore, without loss of generality, we may assume that $d_1 > 0$. In this case the nonempty set $B'_1 \setminus (g_1 + \frac{1}{2} d_1 B)$ is relatively open in $B'_1$ (and therefore in $A_1$). As a first move of player $\Omega$ under the strategy $\omega$ we now take any non-empty relatively open subset $B_1$ of $B'_1$ such that $\overline{B_1} \cap (g_1 + \frac{1}{2} d_1 B) = \varnothing$. This finishes the first induction step.

Suppose that the strategy $\omega$ has already been defined "up to the $n$-th stage", $n \geq 1$, in such a way that each finite $\omega$-play $A_1 \supset B_1 \supset \cdots \supset A_n \supset B_n$ is accompanied by some sets $\{B'_i\}_{i=1}^n$, some points $\{g_i\}_{i=1}^n$ in $C_p(T)$ and some numbers $\{d_i \geq 0\}_{i=1}^n$ so that, for every $i = 1, \ldots, n$, the following properties have place:

a) the points $g_i \in A_i$ and the sets $B_i'$ are the answers of $\Omega$ under the strategy $\omega'$ to the choice $A_i$ of $\Sigma$ in the game $G'(C_p(T))$; in particular, $B_i'$ is a relatively open subset of $A_i$;
b) $B_i = \omega(A_1, \ldots, A_i)$, the answer of $\Omega$ under the strategy $\omega$ to the choices $A_1, \ldots, A_i$ of player $\Sigma$, is a relatively open subset of $B_i'$;
c) $d_i := \inf\{t > 0 : co\{g_1, \ldots, g_i\} + tB \supset B_i'\}$, where $co\{g_1, \ldots, g_i\}$ is the convex hull of the set $\{g_1, \ldots, g_i\}$;
d) The closure $\overline{B_i}$ of $B_i$ in $C_p(T)$ does not intersect the set $co\{g_1, \ldots, g_i\} + \frac{i}{i+1} d_i B$;
e) $\| \cdot \| - diam(B_i) \leq 2(d_i + \frac{1}{i+1})$.

Let $A_{n+1} \subset B_n$ be the next choice of $\Sigma$. Using $\omega'$ player $\Omega$ selects some $g_{n+1} \in A_{n+1}$ and some non-empty relatively open subset $B'_{n+1}$ of $A_{n+1}$. Consider the number

$$d_{n+1} := \inf\{t > 0 : co\{g_1, \ldots, g_{n+1}\} + tB \supset B'_{n+1}\}.$$

Suppose $d_{n+1} = 0$. Then $B'_{n+1}$ is a subset of the finite dimensional compact $co\{g_1, \ldots, g_{n+1}\}$ in which pointwise convergence topology and norm topology coincide. In this case it is easy to define $B_{n+1}$ so that properties d) and e) are fulfilled for $i = n+1$. Moreover, the norm-diameter of $B_{n+1}$ could be taken to be smaller than $\frac{1}{n+1}$. Consider, in the case $d_{n+1} > 0$, the nonempty set

$$B'_{n+1} \setminus \{co\{g_1, \ldots, g_{n+1}\} + \frac{n+1}{n+2} d_{n+1} B\}.$$

It is relatively open in $B'_{n+1}$. Take some nonempty relatively open subset $A$ of $B'_{n+1}$ such that

$$\overline{A} \bigcap \{co\{g_1, \ldots, g_{n+1}\} + \frac{n+1}{n+2} d_{n+1} B\} = \varnothing.$$

Since $co\{g_1, \ldots, g_{n+1}\}$ is compact, there is a finite set $M$ such that

$$co\{g_1, \ldots, g_{n+1}\} \subset M + \frac{1}{n+2} B.$$

Since $A \subset B'_{n+1} \subset co\{g_1, \ldots, g_{n+1}\} + d_{n+1} B$, we have $A \subset M + (d_{n+1} + \frac{1}{n+2})B$. Without loss of generality we can assume that $M$ is a minimal (with respect to its cardinality) finite set with this property. Then, for any $m_0 \in M$, the set

$$B_{n+1} := A \setminus \{\{M \setminus \{m_0\}\} + (d_{n+1} + \frac{1}{n+2})B\} \neq \varnothing$$

is a non-empty relatively open subset of $A$ (and therefore of $A_{n+1}$). This set is taken as the answer of $\Omega$ under the strategy $\omega : B_{n+1} := \omega(A_1, \ldots, A_n, A_{n+1})$. Since $B_{n+1} \subset m_0 + (d_{n+1} + \frac{1}{n+2})B$, we have $\| \cdot \| - diam(B_{n+1}) \leq 2(d_{n+1} + \frac{1}{n+2})$. Thus, condition e) is satisfied.

This, considered as an induction step, completes the construction of the strategy $\omega$.

Let $\{(A_i, B_i)\}_{i \geq 1}$ be an $\omega$-play. Clearly, $\{(A_i, B_i', g_i)\}_{i \geq 1}$, where $B_i'$ and $g_i$ are from the construction of $\omega$, is an $\omega'$-play.

Suppose $\cap_{i\geq 1} A_i = \cap_{i\geq 1} B_i$ is not empty. If $d_n = 0$ for some $n > 0$, then by construction, the diameters of the sets $B_k$ tend to zero. If $d_i > 0$ for every $i \geq 1$, then the sequence $\{g_i\}_{i\geq 1}$ consists of distinct points and, by $(iv)$, it contains a subsequence which is bounded in a separable subset of $C_p(T)$. It follows from Theorem 1.2 that the sequence $\{g_i\}_{i\geq 1}$ contains a subsequence converging in $C_p(T)$ to some function $g_\infty$ which, necessarily, belongs to $\bigcap_{i\geq 1} \overline{B_i}$.

The sequence $\{d_i\}_{i\geq 1}$ of non-negative numbers is non-increasing. Put $d_\infty := \lim_{n\to\infty} d_n$. It suffices to show that $d_\infty = 0$. Suppose that $d_\infty > 0$ and take some positive number $\varepsilon < \frac{1}{2} d_\infty$. Then, by d), we have for every $i \geq 1$

$$g_\infty + \varepsilon B \cap co\{g_1, \dots, g_i\} \subset \{\overline{B_i} + \frac{i}{i+1} d_i B\} \cap co\{g_1, \dots, g_i\} = \varnothing.$$

Therefore $\{g_\infty + \varepsilon B\} \cap \bigcup_{i\geq 1} \{co\{g1, ..., gi\}\} = \varnothing$.

On the other hand, part ii) of Theorem 1.2 says that this is impossible. This contradiction completes the proof of Proposition 2.1 and of Theorem 2.1. □

If the space $T$ has additional properties, then the fragmentability of $C_p(T)$ is implied by even weaker assumptions imposed on the strategy of player $\Omega$. Let us denote by $G''(X)$ a game in the topological space $X$ which differs from $G'(X)$ only by the winning rule: the player $\Omega$ is said to have won the play $\{(A_i, B_i, x_i)\}_{i\geq 1}$ if either the intersection $\cap_{i\geq 1} A_i = \cap_{i\geq 1} B_i$ is empty or, if not empty, there exists some subsequence of the sequence $\{x_i\}_{i\geq 1}$ which is bounded in $X$.

**Theorem 2.2.** *Let $T$ be strongly pseudocompact. Then each of the equivalent conditions $(i)$–$(v)$ (from Theorem 2.1 and Proposition 2.1) is equivalent to the following statement:*

- *(vi) The player $\Omega$ has a winning strategy $\omega''$ in the game $G''(C_p(T))$, i.e. $\omega''$ generates plays $\{(A_i, B_i, g_i)\}_{i\geq 1}$ such that either the intersection $\cap_{i\geq 1} A_i = \cap_{i\geq 1} B_i$ is empty or, otherwise, there exists some subsequence of the sequence $\{g_i\}_{i\geq 1}$ which is bounded in $C_p(T)$.*

*Proof.* The proof is almost identical with the proof of the previous theorem. The only difference is that instead of Theorem 1.2 one uses Theorem 1.1. □

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