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VECTOR-VALUED ALMOST PERIODIC ULTRADISTRIBUTIONS AND THEIR GENERALIZATIONS

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Abstract. In this paper, we introduce the notion of a vector-valued almost periodic ultradistribution and investigate some generalizations of this concept. Albeit contains some original contributions, the paper is primarily intended to review and slightly generalize some known results concerning scalar-valued almost periodic ultradistributions and their generalizations. We contemplate the work of many authors, and transfer several known results on vector-valued almost periodic distributions to vector-valued almost periodic ultradistributions of Beurling and of Roumieu type.

1. Introduction and Preliminaries

The concept of almost periodicity was introduced by Danish mathematician H. Bohr around 1924-1926 and later generalized by many other authors (cf. [1]-[2], [12], [18] and [23] for more details on the subject). Let $I = \mathbb{R}$ or $I = [0, \infty)$, let $(X, \|\cdot\|)$ be a complex Banach space, and let $f : I \to X$ be continuous. Given $\epsilon > 0$, we say that a number $\tau > 0$ is an ϵ-period for $f(\cdot)$ iff $\|f(t+\tau) - f(t)\| \leq \epsilon$, $t \in I$. The set consisting of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f, \epsilon)$. It is said that $f(\cdot)$ is almost periodic iff, for every $\epsilon > 0$, the set $\vartheta(f, \epsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l meets $\vartheta(f, \epsilon)$. By $AP(I : X)$, we denote the vector space consisting of all almost periodic functions; we use the shorthand $C_b(I : X)$ for the space consisting of all bounded continuous functions $I \mapsto X$.

The notion of an almost periodic function has been reconsidered from the point of view of generalized function spaces theory. The notions of bounded and almost periodic distributions have been introduced already in the pioneering papers by L. Schwartz (see e.g. [28]), who analyzed only

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scalar-valued case. The bounded and almost periodic distributions with values in general Banach spaces have been investigated for the first time by I. Cioranescu in [10]: Let $\mathcal{D}_{L^1}$ denote the vector space consisting of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$ satisfying that for each number $j \in \mathbb{N}_0$ we have $f^{(j)} \in L^1(\mathbb{R})$. The Fréchet topology on $\mathcal{D}_{L^1}$ is induced by the following system of seminorms

$$\|f\|_k := \sum_{j=0}^{k} \big\|f^{(j)}\big\|_{L^1(\mathbb{R})}, \quad f \in \mathcal{D}_{L^1} \quad \big(k \in \mathbb{N}\big).$$

A continuous linear mapping $f : \mathcal{D}_{L^1} \to X$ is said to be the bounded X-valued distribution. The space of such distributions is usually denoted by $B'(X)$; equipped with the strong topology, $B'(X)$ becomes a complete locally convex space. Our first observation is given as follows: If $f : \mathcal{D}_{L^1} \to X$ is a bounded X-valued distribution, then there exist $c > 0$ and $k \in \mathbb{N}$ such that

$$\|f(\varphi)\| \leq c \sum_{j=0}^{k} \int_{-\infty}^{\infty} |\varphi^{(j)}(x)|\, dx = c \sum_{j=0}^{k} \int_{-\infty}^{\infty} \tfrac{1}{x^2+1}[(x^2+1)|\varphi^{(j)}(x)|]\, dx \leq$$
$$\leq c \int_{-\infty}^{\infty} \tfrac{dx}{x^2+1} \cdot \sum_{j=0}^{k} \big\|(\cdot^2+1)\varphi^{(j)}(\cdot)\big\|_{\infty}, \quad \varphi \in \mathcal{S},$$

so that $f_{|\mathcal{S}} : \mathcal{S} \to X$ is a tempered X-valued distribution (here and hereafter, the space of rapidly decreasing functions $\mathcal{S}$ carries the usual Fréchet topology; the symbol $\mathcal{D}$ denotes the Schwartz space of test functions and $\mathcal{D}'(X)$ denotes the space of all continuous linear mappings $\mathcal{D} \to X$, equipped with the strong topology). Following L. Schwartz [28] and I. Cioranescu [10], we say that a bounded X-valued distribution $f \in B'(X)$ is almost periodic iff there exists a sequence of X-valued trigonometric polynomials converging to $f(\cdot)$ in $B'(X)$. If $B'_{ap}(X)$ denotes the vector space consisting of all almost periodic X-valued distributions, then $AP(\mathbb{R} : X)$ is dense in $B'_{ap}(X)$ by definition. Furthermore, it is well known that an element $f \in \mathcal{D}'(X)$ belongs to $B'(X)$, resp., $B'_{ap}(X)$ iff there is an integer $k \in \mathbb{N}_0$ such that $f = \sum_{j=0}^{k} f_j^{(j)}$, where $f_j \in C_b(\mathbb{R} : X)$, resp., $f_j \in AP(\mathbb{R} : X)$, for $0 \leq j \leq k$ iff for any $\varphi \in \mathcal{D}$, we have $f * \varphi \in C_b(\mathbb{R} : X)$, resp., $f * \varphi \in AP(\mathbb{R} : X)$ iff the set of all translations of $f(\cdot)$, defined as usually, is bounded in $\mathcal{D}'(X)$, resp., relatively compact in $B'(X)$. The spaces of vector-valued almost periodic distributions are systematically analyzed in a series of research papers by B. Basit and H. Güenzler (see e.g. [3]-[4]). Here we would like to mention that they have proved [4] that any regular vector-valued distribution $\varphi \mapsto \int_{-\infty}^{\infty} f(t)\varphi(t)\, dt$, $\varphi \in \mathcal{D}$, where $f : \mathbb{R} \to X$ is a Stepanov p-almost periodic function for some $p \in [1, \infty)$, is almost periodic, as well as that, for every $p \in [1, \infty)$, there exists a scalar-valued infinitely differentiable Weyl p-almost periodic function $f(\cdot)$ such that the regular distribution given by the above formula is not almost periodic (cf. [23] for the notion).

Within the Komatsu theory of ultradistributions, the notion of a scalar-valued almost periodic ultradistribution has been introduced by I. Cioranescu [11]. In her approach, the corresponding sequence (M_p) always satisfies the conditions (M.1), (M.2) and (M.3). The results from [11] have been reconsidered and slightly generalized by M. C. Gómez-Collado [16] and C. Fernández, A. Galbis, M. C. Gómez-Collado [17], within the theory of ω-ultradistributions (cf. R. W. Braun, R. Meise, B. A. Taylor [8]). In this paper, we basically follow Komatsu's approach, with the sequence (M_p) satisfying the conditions (M.1), (M.2) and (M.3'); any use of the condition (M.3) will be explicitly emphasized.

To the best knowledge of the author, the notion of a vector-valued almost periodic ultradistribution has not been yet introduced in the existing literature, even in the case that the sequence (M_p) satisfies the condition (M.3). And, more to the point, the notion of a scalar-valued almost periodic ultradistribution, introduced in this paper, seems to be new in the case that (M_p) does not satisfy the condition (M.3). Regarding scalar-valued almost periodic ultradistributions as boundary values of harmonic almost periodic functions, we would like to mention that I. Cioranescu [11] has proved that, for any such ultradistribution, we can find a harmonic function $u(x, y)$ in the right-half plane such that, for every $x > 0$ the mapping $y \mapsto u(x, y)$, $y \in \mathbb{R}$ is almost periodic and that $\lim_{x\to 0+} u(x, y) = f$ in the ultradistributional sense. The main aim of this paper is, actually, to reexamine the structural results proved in [11], [4] and [16]. The concept of almost automorphic ultradistributions will be introduced and analyzed in our follow-up research with S. Pilipović and D. Velinov [24] (cf. C. Bouzar, M. T. Khalladi, F. Z. Tchouar [5] for the notion of an almost automorphic Colombeau generalized function, C. Bouzar, Z. Tchouar [6] for the notion of an almost automorphic distribution, C. Bouzar, M. T. Khalladi [7] for the notion of an almost periodic Colombeau generalized function, and M. F. Hasler [19] for the notion of a Bloch-periodic Colombeau generalized function).

Before recollecting some known facts about vector-valued ultradistributions, we would like to draw the attention of our readers to the excellent survey of results [29] by V. Valmorin, concerning periodic generalized functions and their applications.

1.1. **Vector-valued ultradistributions.** In anything that follows, it will be assumed that (M_p) is a sequence of positive real numbers satisfying $M_0 = 1$ and the following conditions:

(M.1): $M_p^2 \leq M_{p+1}M_{p-1},\ \ p \in \mathbb{N}$,

(M.2): $M_p \leq AH^p \sup_{0\leq i\leq p} M_i M_{p-i},\ \ p \in \mathbb{N},$ for some $A,\ H > 1$,

(M.3'): $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$.

Any employment of the condition
(M.3): $\sup_{p\in\mathbb{N}}\sum_{q=p+1}^{\infty}\frac{M_{q-1}M_{p+1}}{pM_pM_q}<\infty$,
which is slightly stronger than (M.3′), will be explicitly emphasized.

Let $s>1$. The Gevrey sequence $(p!^s)$ satisfies the above conditions. The associated function of (M_p) is defined by $M(\rho):=\sup_{p\in\mathbb{N}}\ln\frac{\rho^p}{M_p}$, $\rho>0$; $M(0):=0$. If $\lambda\in\mathbb{C}$, set $M(\lambda):=M(|\lambda|)$. Define $m_p:=\frac{M_p}{M_{p-1}}$, $p\in\mathbb{N}$.

The space of Beurling, resp., Roumieu ultradifferentiable functions, is defined by $\mathcal{D}^{(M_p)}:=\text{indlim}_{K\Subset\Subset\mathbb{R}}\mathcal{D}_K^{(M_p)}$, resp., $\mathcal{D}^{\{M_p\}}:=\text{indlim}_{K\Subset\Subset\mathbb{R}}\mathcal{D}_K^{\{M_p\}}$, where $\mathcal{D}_K^{(M_p)}:=\text{projlim}_{h\to\infty}\mathcal{D}_K^{M_p,h}$, resp., $\mathcal{D}_K^{\{M_p\}}:=\text{indlim}_{h\to 0}\mathcal{D}_K^{M_p,h}$,

$$\mathcal{D}_K^{M_p,h}:=\bigl\{\phi\in C^\infty(\mathbb{R}):\text{supp}\phi\subseteq K,\ \|\phi\|_{M_p,h,K}<\infty\bigr\}$$

and

$$\|\phi\|_{M_p,h,K}:=\sup\Biggl\{\frac{h^p|\phi^{(p)}(t)|}{M_p}:t\in K,\ p\in\mathbb{N}_0\Biggr\}.$$

In the sequel, the asterisk $*$ is used to designate both, the Beurling case (M_p) or the Roumieu case $\{M_p\}$. The space consisted of all continuous linear functions from $\mathcal{D}^*$ into X, denoted by $\mathcal{D}'^*(X):=L(\mathcal{D}^*:X)$, is said to be the space of all X-valued ultradistributions of $*$-class.

Recall [20], an entire function of the form $P(\lambda)=\sum_{p=0}^{\infty}a_p\lambda^p$, $\lambda\in\mathbb{C}$, is of class (M_p), resp., of class $\{M_p\}$, if there exist $l>0$ and $C>0$, resp., for every $l>0$ there exists a constant $C>0$, such that $|a_p|\leq Cl^p/M_p$, $p\in\mathbb{N}$. The corresponding ultradifferential operator $P(D)=\sum_{p=0}^{\infty}a_pD^p$ is of class (M_p), resp., of class $\{M_p\}$. We introduce the topology of above spaces as well as the convolution of scalar valued ultradistributions (ultradifferentiable functions) in the same way as in the case of corresponding distribution spaces ([20]). The convolution of Banach space valued ultradistributions and scalar-valued ultradifferentiable functions will be taken in the sense of considerations given on page 685 of [22]. Let us recall that for any $f\in\mathcal{D}'^*(X)$ and $\varphi\in\mathcal{D}^*$ we have $f*\varphi\in\mathcal{E}^*(X)$ as well as that the linear mapping $\varphi\mapsto\cdot*\varphi:\mathcal{D}'^*(X)\to\mathcal{E}^*(X)$ is continuous. Here, the space $\mathcal{E}^*(X)$ is defined as it has been done on page 678 of [22]. The convolution of an X-valued ultradistribution $f(\cdot)$ and an element $g\in\mathcal{E}'^*$, defined by the identity [22, (4.9)], is an X-valued ultradistribution and the mapping $g*\cdot:\mathcal{D}'^*(X)\to\mathcal{D}'^*(X)$ is continuous. Set $\langle T_h,\varphi\rangle:=\langle T,\varphi(\cdot-h)\rangle$, $T\in\mathcal{D}'^*(X)$, $h>0$.

If (M_p) satisfies (M.1), (M.2) and (M.3), then

$$P_l(x)=\bigl(1+x^2\bigr)\prod_{p\in\mathbb{N}}\Biggl(1+\frac{x^2}{l^2m_p^2}\Biggr),$$

resp.

$$P_{r_p}(x) = \left(1 + x^2\right) \prod_{p\in\mathbb{N}} \left(1 + \frac{x^2}{m_p^2 r_p^2}\right),$$

defines an ultradifferential operator of class (M_p), resp., of class $\{M_p\}$. Here, $(r_p) \in \mathrm{R}$, where R denotes the family of all sequences of positive real numbers tending to infinity.

The following spaces of tempered ultradistributions of Beurling, resp., Roumieu type, are defined by S. Pilipović [27] as duals of the corresponding test spaces

$$\mathcal{S}^{(M_p)} := \mathrm{projlim}_{h\to\infty} \mathcal{S}^{M_p,h}, \quad \text{resp.,} \quad \mathcal{S}^{\{M_p\}} := \mathrm{indlim}_{h\to 0} \mathcal{S}^{M_p,h},$$

where

$$\mathcal{S}^{M_p,h} := \left\{\phi \in C^\infty(\mathbb{R}) : \|\phi\|_{M_p,h} < \infty\right\} \quad (h > 0),$$

$$\|\phi\|_{M_p,h} := \sup\left\{\frac{h^{\alpha+\beta}}{M_\alpha M_\beta}(1+t^2)^{\beta/2}|\phi^{(\alpha)}(t)| : t \in \mathbb{R},\ \alpha,\ \beta \in \mathbb{N}_0\right\}.$$

A continuous linear mapping $\mathcal{S}^{(M_p)} \to X$, resp., $\mathcal{S}^{\{M_p\}} \to X$, is said to be an X-valued tempered ultradistribution of Beurling, resp., Roumieu type. The space consisted of all vector-valued tempered ultradistributions of Beurling, resp., Roumieu type, will be denoted by $\mathcal{S}'^{(M_p)}(X)$, resp. $\mathcal{S}'^{\{M_p\}}(X)$; the common abbreviation will be $\mathcal{S}'^{*}(X)$. It is well known that $\mathcal{S}'^{(M_p)}(X) \subseteq \mathcal{D}'^{(M_p)}(X)$, resp. $\mathcal{S}'^{\{M_p\}}(X) \subseteq \mathcal{D}'^{\{M_p\}}(X)$.

2. Almost periodicity of vector-valued ultradistributions

For any $h > 0$, we define

$$\mathcal{D}_{L^1}\left((M_p), h\right) := \left\{f \in \mathcal{D}_{L^1}\ ;\ \|f\|_{1,h} := \sup_{p\in\mathbb{N}_0} \frac{h^p \|f^{(p)}\|_1}{M_p} < \infty\right\}.$$

Then $(\mathcal{D}_{L^1}((M_p), h), \|\cdot\|_{1,h})$ is a Banach space and the space of all X-valued bounded Beurling ultradistributions of class (M_p), resp., X-valued bounded Roumieu ultradistributions of class $\{M_p\}$, is defined as the space consisting of all linear continuous mappings from $\mathcal{D}_{L^1}((M_p))$, resp., $\mathcal{D}_{L^1}(\{M_p\})$, into X, where

$$\mathcal{D}_{L^1}\left((M_p)\right) := \mathrm{projlim}_{h\to+\infty} \mathcal{D}_{L^1}\left((M_p), h\right),$$

resp.,

$$\mathcal{D}_{L^1}\left(\{M_p\}\right) := \mathrm{indlim}_{h\to 0+} \mathcal{D}_{L^1}\left((M_p), h\right).$$

These spaces, carrying the strong topologies, will be shortly denoted by $\mathcal{D}'_{L^1}((M_p) : X)$, resp., $\mathcal{D}'_{L^1}(\{M_p\} : X)$. It is well known that $\mathcal{D}^{(M_p)}$, resp. $\mathcal{D}^{\{M_p\}}$, is a dense subspace of $\mathcal{D}_{L^1}((M_p))$, resp., $\mathcal{D}_{L^1}(\{M_p\})$, as well as that $\mathcal{D}_{L^1}((M_p)) \subseteq \mathcal{D}_{L^1}(\{M_p\})$ (see [9]). Since $\|\varphi\|_{1,h} \leq \|\varphi\|_{M_p,h}$ for any $\varphi \in \mathcal{S}^{(M_p)}$ and $h > 0$, we have that $\mathcal{S}^{(M_p)}$, resp. $\mathcal{S}^{\{M_p\}}$, is a dense subspace of $\mathcal{D}_{L^1}((M_p))$, resp., $\mathcal{D}_{L^1}(\{M_p\})$, and that $f_{|\mathcal{S}^{(M_p)}} : \mathcal{S}^{(M_p)} \to X$, resp., $f_{|\mathcal{S}^{\{M_p\}}} : \mathcal{S}^{\{M_p\}} \to X$, is a tempered X-valued ultradistribution of class (M_p), resp., of class $\{M_p\}$.

Following I. Cioranescu [11], we say that a bounded X-valued ultradistribution $f \in \mathcal{D}'_{L^1}((M_p) : X)$, resp., $f \in \mathcal{D}'_{L^1}(\{M_p\} : X)$, is almost periodic of Beurling class (M_p), resp., almost periodic of Roumeiu class $\{M_p\}$, iff there exists a sequence of X-valued trigonometric polynomials converging to $f(\cdot)$ in $\mathcal{D}'_{L^1}((M_p) : X)$, resp., $\mathcal{D}'_{L^1}(\{M_p\} : X)$. In the case that (M_p) satisfies the conditions (M.1), (M.2) and (M.3), the space of all bounded scalar-valued ultradistributions of Beurling class has been characterized in [11, Theorem 1] and the space of all almost periodic scalar-valued ultradistributions of Beurling class has been characterized in [11, Theorem 2]; the condition (M.3) is essentially employed in the proof of [11, Lemma 2], which is no longer true if we assume only the condition (M.3') and which is a fundamental tool for proving the implications [11, Theorem 1, (iii) ⇒ (iv)] and [11, Theorem 2, (iv) ⇒ (ii)]. The assertion of [11, Lemma 1] continues to hold if the condition (M.3) is disregarded, in both Beurling and Roumieu case, that is: Suppose that $P(D) = \sum_{p=0}^{\infty} a_p D^p$ is an ultradifferential operator of class (M_p), resp., of class $\{M_p\}$. Then the induced mapping $\mathbf{P}_B(D) : \mathcal{D}_{L^1}((M_p)) \to \mathcal{D}_{L^1}((M_p))$, resp., $\mathbf{P}_R(D) : \mathcal{D}_{L^1}(\{M_p\}) \to \mathcal{D}_{L^1}(\{M_p\})$ is linear and continuous.

The following theorem gives some new insights into the assertion of [11, Theorem 1]:

Theorem 1. *Let* (M_p) *satisfy the conditions* (M.1), (M.2) *and* (M.3'), *and let* $T \in \mathcal{D}'^*(X)$. *Consider the following assertions:*

(i) There exists an ultradifferential operator $P(D) = \sum_{p=0}^{\infty} a_p D^p$ *of class* (M_p), *resp., of class* $\{M_p\}$, *and functions* $f,\ g \in C_b(\mathbb{R} : X)$ *such that* $T = P(D)f + g$, *i.e.,*

$$\langle T, \varphi\rangle = \sum_{p=0}^{\infty} (-1)^p a_p \int_{-\infty}^{\infty} f(t)\varphi^{(p)}(t)\,dt + \int_{-\infty}^{\infty} g(t)\varphi(t)\,dt, \qquad (1)$$

for all $\varphi \in \mathcal{D}_{L^1}((M_p))$, *resp.,* $\varphi \in \mathcal{D}_{L^1}(\{M_p\})$.

(ii) We have $T \in \mathcal{D}'_{L^1}((M_p) : X)$, *resp.,* $\mathcal{D}'_{L^1}(\{M_p\} : X)$.

(iii) For every $\varphi \in \mathcal{D}^*$, *we have* $T * \varphi \in C_b(\mathbb{R} : X)$; *furthermore, if* $B \subseteq \mathcal{D}^*$ *is bounded, then there exists a finite constant* $M \geq 1$ *such that* $||T * \varphi||_\infty \leq M$, $\varphi \in B$.

(iv) For each compact set $K \subseteq \mathbb{R}$ *there exists* $h > 0$ *in the Beurling case, resp., for each compact set* $K \subseteq \mathbb{R}$ *and for every* $h > 0$ *in the Roumieu case, we have* $T * \varphi \in C_b(\mathbb{R} : X)$, $\varphi \in \mathcal{D}_K^{M_p,h}$.

Then we have (i) ⇒ (ii) ⇒ (iii) ⇒ (iv).

Proof. The implications (ii) ⇒ (iii) ⇒ (iv) follow from the proof of abovementioned theorem and elementary facts about topological properties of vector-valued ultradistributions. We will give a direct proof of the assertion (i) $\Rightarrow$ (ii) here, in which we do not use the condition (M.2); for the sake of brevity, we examine only the Roumieu case. We need to prove that for each $h>0$ the mapping $T:\mathcal{D}_{L^1}((M_p),h)\rightarrow X$ is continuous. By our assumption, for every $l\in(0,h)$, there exists $c_l>0$ such that $|a_p|\leq c_l l^p/M_p$, $p\geq 0$. Hence, for every $\varphi\in\mathcal{D}_{L^1}((M_p),h)$, we have:

$$\begin{aligned}\big\|\langle T,\varphi\rangle\big\| &\leq \sum_{p=0}^{\infty}\|f\|_\infty|a_p|\big\|\varphi^{(p)}\big\|_1+\|g\|_\infty\big\|\varphi\big\|_1\\ &\leq \sum_{p=0}^{\infty}\|f\|_\infty\frac{|c_l|}{M_p}l^ph^{-p}\|\varphi\|_{1,h}M_p+\|g\|_\infty\big\|\varphi\big\|_1\\ &\leq c_l\|f\|_\infty\|\varphi\|_{1,h}\frac{h}{h-l}+\|g\|_\infty\big\|\varphi\big\|_1.\end{aligned}$$

This, in turn, yields (ii). □

Remark 1. (i) *Assume that (M_p) additionally satisfies (M.3). Then the assertions (i)-(iv) are mutually equivalent for the Beurling class* [11] *and there exists $l>0$ such that the choice $P(D)=P_l(D)$ is possible in (i). It is not clear whether this statement holds for the Roumieu class, with the operator $P(D)=P_{r_p}(D)$ and some $(r_p)\in$ R; see also* [11, Lemma 2] *and* [9, Lemma 3.1.1(ii)].

(ii) *The assertion of* [16, Theorem 3.2] *continues to hold in vector-valued case.*

Concerning [11, Theorem 2], the following result should be stated in vector-valued case (in the Beurling case, the implication (iii) $\Rightarrow$ (iv) follows from the fact that the equation [26, (13)] holds in vector-valued case and the proof of corresponding implication [11, (iii) $\Rightarrow$ (iv), Theorem 2]; in the Roumieu case, the statement follows directly):

Theorem 2. *Let (M_p) satisfy the conditions* (M.1), (M.2) *and* (M.3'), *and let $T\in\mathcal{D}'_{L^1}((M_p):X)$, resp., $T\in\mathcal{D}'_{L^1}(\{M_p\}:X)$. Consider the following assertions:*

(i) There exists an ultradifferential operator $P(D)=\sum_{p=0}^{\infty}a_pD^p$ of class (M_p), resp., of class $\{M_p\}$, and functions $f,\ g\in AP(\mathbb{R}:X)$ such that $T=P(D)f+g$, i.e., (1) *holds for all $\varphi\in\mathcal{D}_{L^1}((M_p))$, resp., $\varphi\in\mathcal{D}_{L^1}(\{M_p\})$.*

(ii) T is almost periodic.

(iii) For every $\varphi\in\mathcal{D}^*$, we have $T*\varphi\in AP(\mathbb{R}:X)$.

(iv) There exists $h>0$ such that for each compact set $K\subseteq\mathbb{R}$, in the Beurling case, resp., for each compact set $K\subseteq\mathbb{R}$ and for each $h>0$, in the Roumieu case, the following holds $T*\varphi\in AP(\mathbb{R}:X)$, $\varphi\in\mathcal{D}_K^{M_p,h}$.

Then we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv)*.*

Remark 2. (i) *Assume that (M_p) additionally satisfies (M.3). Then the above assertions are equivalent for the Beurling class, when there exists $l > 0$ such that the choice $P(D) = P_l(D)$ is possible in (i), and it is not clear whether these assertions are equivalent for the Roumieu class, with the operator $P(D) = P_{r_p}(D)$ and some $(r_p) \in$* R.

(ii) *It is worth noting that* [16, Theorem 4.2] *continues to hold in vector-valued case. Consider the following assertion:*

(ii)' *The set of all translations $\{T_h : h \in \mathbb{R}\}$ is relatively compact in $\mathcal{D}'_{L^1}((M_p) : X)$, resp., $\mathcal{D}'_{L^1}(\{M_p\} : X)$.*

Using the same arguments as in the proof of [16, Theorem 4.2]*, we can deduce that (i) $\Rightarrow$ (ii)' $\Rightarrow$ (iii).*

Let us introduce the following space

$$\mathcal{E}^*_{AP}(X) := \left\{ \phi \in \mathcal{E}^*(X) : \phi^{(i)} \in AP(\mathbb{R} : X) \text{ for all } i \in \mathbb{N}_0 \right\}.$$

Then $\mathcal{E}^*_{AP}(X) \subseteq \mathcal{D}'^*_{L^1}(X)$ and, due to the fact that the first derivative of a differentiable almost periodic function is almost periodic iff it is uniformly continuous [23] and the proof of [6, Proposition 5(i)], $\mathcal{E}^*_{AP}(X) = \mathcal{E}^*(X) \cap AP(\mathbb{R} : X)$. Furthermore, $\mathcal{E}^*_{AP}(X) * L^1(\mathbb{R}) \subseteq \mathcal{E}^*_{AP}(X)$ and $\mathcal{E}^*_{AA}(X)$ is the space consisted exactly of those elements $f(\cdot)$ from $\mathcal{E}^*(X)$ for which $f * \varphi \in AP(\mathbb{R} : X)$, $\varphi \in \mathcal{D}^*$; see e.g. the proof of [6, Corollary 1] given in distribution case, for almost automorphy.

Consider now the following statement:

(ii)": $T \in \mathcal{D}'^*_{L^1}((M_p) : X)$, resp. $T \in \mathcal{D}'^*_{L^1}(\{M_p\} : X)$, and there exists a sequence (ϕ_n) in $\mathcal{E}^*_{AP}(X)$ such that $\lim_{n\to\infty} \phi_n = T$ for the topology of $\mathcal{D}'_{L^1}((M_p) : X)$, resp. $\mathcal{D}'_{L^1}(\{M_p\} : X)$.

Lemma 1. *Let (M_p) satisfy the conditions* (M.1), (M.2) *and* (M.3')*, and let $T \in \mathcal{D}'_{L^1}((M_p) : X)$, resp., $T \in \mathcal{D}'_{L^1}(\{M_p\} : X)$. Then* (ii)" $\Leftrightarrow$ (iii)*, with* (iii) *being the same as in the formulation of* Theorem 2*.*

Proof. The proof of implication (ii)' $\Rightarrow$ (iii) can be deduced as in distribution case (see the proof of [6, Proposition 7]), while the implication (iii) $\Rightarrow$ (ii') can be proved in the following way. Let $\rho \in \mathcal{D}^*$, $\text{supp}(\rho) \subseteq [0, 1]$ and $T_n := T * \rho_n$ $(n \in \mathbb{N})$. Since $\mathcal{E}^*_{AP}(X) = \mathcal{E}^*(X) \cap AP(\mathbb{R} : X)$, (ii) yields that $T_n \in \mathcal{E}^*_{AP}(X)$ for all $n \in \mathbb{N}$. Then it suffices to show that $\lim_{n\to\infty} T_n = T$ in $\mathcal{D}'^*_{L^1}(X)$. For the sake of brevity, we will consider only the Roumieu class. Let $h > 0$ be fixed. Then there exists $c > 0$ such that $\|\langle T, \varphi\rangle\| \leq c\|\varphi\|_{1,h}$, $\varphi \in \mathcal{D}_{L^1}((M_p), h)$. Furthermore, the estimate $\|[\check{\rho}_n * \varphi - \varphi]^{(p)}\|_{L^1} \leq (1/n)\|\varphi^{(p+1)}\|_{L^1}$ $(n \in \mathbb{N},\ p \in \mathbb{N}_0)$ holds good on account of the proof of [6, Proposition 7], so that:

$$\begin{aligned}\Big\|\big\langle T * \rho_n - T, \varphi\big\rangle\Big\| &= \Big\|\big\langle T, \check{\rho}_n * \varphi - \varphi\big\rangle\Big\| \\ &\le c \sup_{p\ge 0} \frac{h^p \Big\|\big[\check{\rho}_n * \varphi - \varphi\big]^{(p)}\Big\|_{L_1}}{M_p} \\ &\le \frac{c}{n} \sup_{p\ge 0} \frac{h^p \big\|\varphi^{(p+1)}\big\|_{L^1}}{M_p} \\ &\le \frac{c}{nh} A M_1 \sup_{p\ge 0} \frac{(hH)^{p+1} \|\varphi^{(p+1)}\|_{L^1}}{M_{p+1}} \\ &\le \frac{c}{nh} A M_1 \|\varphi\|_{1,h}, \quad \varphi \in \mathcal{D}_{L^1}((M_p), h),\end{aligned}$$

which simply completes the proof by using some elementary topological properties of spaces $\mathcal{D}^*_{L^1}$ and $\mathcal{D}'_{L^1}((M_p) : X)$, resp. $\mathcal{D}'_{L^1}(\{M_p\} : X)$. ☐

Now we can prove the following result:

Theorem 3. *Let* (M_p) *satisfy the conditions* (M.1), (M.2) *and* (M.3'), *and let* $T \in \mathcal{D}'_{L^1}((M_p) : X)$, *resp.,* $T \in \mathcal{D}'_{L^1}(\{M_p\} : X)$. *Consider the assertions* (ii), (iii) *and* (iv) *stated in the formulation of* Theorem 2. *Then we have* (ii) ⇔ (ii)" ⇔ (iii) ⇔ (iv).

Proof. By Theorem 2 and Lemma 1, we only need to prove that (ii) implies (ii)" and that (iv) implies (iii). The equivalence of (iv) and (iii) follows directly in Roumieu case, by definition, while the implication (iv) ⇔ (iii) in Beurling case follows from the fact that $\mathcal{D}_K^{(M_p)} = \bigcap_{h>0} \mathcal{D}_K^{M_p,h}$. To prove that (ii)" implies (ii), it suffices to observe that any function $f \in AP(\mathbb{R} : X)$, and therefore any function $f \in \mathcal{E}^*_{AP}(X)$, can be uniformly approximated by trigonometric polynomials. ☐

For any $f \in AP(\mathbb{R} : X)$, we define

$$M(f) := \lim_{t\to\infty} \frac{1}{t} \int_0^t f(s)\, ds.$$

Let $T \in \mathcal{D}'_{L^1}((M_p) : X)$, resp., $T \in \mathcal{D}'_{L^1}(\{M_p\} : X)$ be almost periodic. Following the analyses from [10] and [16], we define the Bohr-Fourier coefficients $a_\lambda(T)$ of T by

$$M(f) := \frac{M(T * \varphi)}{\int_{-\infty}^{\infty} \varphi(s)\, ds},$$

where $\varphi \in \mathcal{D}'_{L^1}((M_p) : X)$, resp., $\varphi \in \mathcal{D}'_{L^1}(\{M_p\} : X)$ is fixed and satisfies $\int_{-\infty}^{\infty} \varphi(s)\, ds \neq 0$. Then $T = 0$ iff $a_\lambda(T) = 0$, $\lambda \in \mathbb{R}$; as in the case of almost periodic functions, we have that the spectrum of T, consisted of all real numbers λ for which $a_\lambda(T) \neq 0$, is at most countable (see [16, Proposition 4.5]). The assertions of [16, Theorem 4.6, Proposition 4.7], regarding Bochner-Féjer summation methods for ultradistributions, continue to hold in vector-valued case and it is not difficult to see that these statements hold for Komatsu's vector-valued almost ultradistributions, even in the case that the condition (M.3) is not satisfied.

3. Generalizations of vector-valued almost periodic ultradistributions

Let $\mathbb{A} \subseteq L^1_{loc}(\mathbb{R} : X)$. It is said that $\mathbb{A}$ satisfies the condition $(\triangle)$ iff for every $f \in L^1_{loc}(\mathbb{R} : X)$ with the property that $\triangle_s f := f(\cdot + s) - f(\cdot) \in \mathbb{A}$ for all $s > 0$, we have $f - M_h f \in \mathbb{A}$, $h > 0$, where

$$M_h f(\cdot) := \frac{1}{h} \int_0^h f(\cdot + s)\, ds, \quad h > 0.$$

If the preassumption $\triangle_s f \in \mathbb{A}$, $s > 0$ implies $f - M_1 f \in \mathbb{A}$, for any $f \in L^1_{loc}(\mathbb{R} : X)$, then it is said that $\mathbb{A}$ satisfies the condition $(\triangle_1)$. For more details about the importance of Doss conditions $(\triangle)$ and $(\triangle_1)$ in the theory of almost periodic vector-valued functions, the reader may consult [3]-[4] and references cited therein.

Define

$$\mathrm{M}(\mathbb{A}) := \Big\{ f \in L^1_{loc}(\mathbb{R} : X) : M_h f \in \mathbb{A} \text{ for all } h > 0 \Big\}.$$

Following B. Basit and H. Güenzler [3]-[4], we introduce the following space of vector-valued ultradistributions:

$$\mathcal{D}'^*_{\mathbb{A}}(X) := \Big\{ T \in \mathcal{D}'^*(X) : T * \varphi \in \mathbb{A} \text{ for all } \varphi \in \mathcal{D}^* \Big\}.$$

If $\mathrm{A} \subseteq \mathcal{D}'^*(X)$, then we similarly define the space

$$\mathcal{D}'^*_{\mathrm{A}}(X) := \Big\{ T \in \mathcal{D}'^*(X) : T * \varphi \in \mathrm{A} \text{ for all } \varphi \in \mathcal{D}^* \Big\}.$$

Note that $\mathcal{D}'^*_{\mathrm{A}}(X) = \mathcal{D}'^*_{\mathbb{A}}(X) = \mathcal{D}'^*_{\mathbb{A}_\infty}(X)$, where $\mathbb{A} = \mathrm{A} \cap L^1_{loc}(\mathbb{R} : X)$ and $\mathbb{A}_\infty = \mathrm{A} \cap C^\infty(\mathbb{R} : X)$. By Theorem 2, we have that the space consisting of almost periodic vector-valued ultradistributions is contained in the space $\mathcal{D}'^*_{\mathbb{A}}(X)$, where $\mathbb{A} = AP(\mathbb{R} : X)$.

Let $s_h := (1/h)\chi_{(-h,0)}$ $(h > 0)$ and $s_h := ((-1)/h)\chi_{(0,-h)}$ $(h < 0)$. Then, for any $f \in L^1_{loc}(\mathbb{R} : X)$, we have $M_h f = f * s_h$, $h > 0$. For vector-valued ultradistributions, we set $\langle T_s, \varphi\rangle := \langle T, \varphi(\cdot - s)\rangle$, $\varphi \in \mathcal{D}^*$ $(s \in \mathbb{R}$, $T \in \mathcal{D}'^*(X))$,

$$\widetilde{M_h} T := T * s_h, \quad h \neq 0, \ \ T \in \mathcal{D}'^*(X)$$

and, for any subset A of $\mathcal{D}'^*(X)$,

$$\widetilde{\mathcal{M}}(\mathrm{A}) := \Biggl\{ T \in \mathcal{D}'^*(X) : \widetilde{M_h}T \in \mathrm{A} \text{ for all } h > 0 \Biggr\}.$$

It is said that $T \in \mathcal{D}'^*(X)$ satisfies $(\triangle)$ iff $\triangle_s T := T_{-s} - T \in \mathcal{D}'^*_{\mathbb{A}}(X)$ for all $s > 0$ implies $T - \widetilde{M_h}T \in \mathcal{D}'^*_{\mathbb{A}}(X)$ for all $h > 0$; if the assumption $\triangle_s T \in \mathcal{D}'^*_{\mathbb{A}}(X)$ for all $s > 0$ implies $T - \widetilde{M_1}T \in \mathcal{D}'^*_{\mathbb{A}}(X)$, then it is said that T satisfies $(\triangle_1)$.

The space $\mathcal{D}'^*_{\mathrm{A}}(X)$ is closed under the action of any ultradifferential operator $P(D)$ of $*$-class because, due to [20, Theorem 2.12], we have:

$$P(D)T * \varphi = T * P(D)\varphi \in \mathrm{A}, \quad \varphi \in \mathcal{D}^*.$$

By [4, Theorem 2.10], we have that the closedness of A under addition implies that, for any vector-valued distribution $T \in \mathcal{D}'(X)$, we have:

$$T * \varphi * \psi \in \mathrm{A}, \quad \varphi,\ \psi \in \mathcal{D} \ \Rightarrow T * \varphi \in \mathrm{A}, \quad \varphi \in \mathcal{D}. \tag{2}$$

To the best knowledge of the author, it is still unknown whether an ultradifferentiable function of $*$-class can be written as a finite sum of functions like $\varphi * \psi$, where $\varphi,\ \psi$ are ultradifferentiable functions of $*$-class. Because of that, in the present situation, we are not able to say whether (2) holds in ultradistribution case, if A is only closed under addition. But, we would like to note that the closedness of set $\mathrm{A} \cap C(\mathbb{R} : X)$ under uniform convergence over $\mathbb{R}$ also implies that, for any vector-valued distribution $T \in \mathcal{D}'(X)$, we have (2). The proof can be given as in ultradistribution case, where we have the following statement:

Proposition 1. *Let* $\mathrm{A} \cap C(\mathbb{R} : X)$ *be closed under uniform convergence over* $\mathbb{R}$. *Then, for any vector-valued ultradistribution* $T \in \mathcal{D}'^*(X)$, *we have:*

$$T * \varphi * \psi \in \mathrm{A}, \quad \varphi,\ \psi \in \mathcal{D}^* \ \Rightarrow T * \varphi \in \mathrm{A}, \quad \varphi \in \mathcal{D}^*. \tag{3}$$

Proof. It is well-known that there is a function $\rho \in \mathcal{D}^*_{[0,1]}$ such that

$$\int_{-\infty}^{\infty} \rho(t)\, dt = 1.$$

Set $\rho_n(t) := n\rho(nt)$, $t \in \mathbb{R}$, $n \in \mathbb{N}$. It suffices to show that, for every $\varphi \in \mathcal{D}^*$, we have

$$\lim_{n \to +\infty} \Bigl(T * \check{\rho_n} * \varphi \Bigr)(x) = \Bigl(T * \varphi \Bigr)(x),$$

uniformly on $\mathbb{R}$ (although this basically follows from the proof of implication [11, (iii) $\Rightarrow$ (iv), Theorem 2], we want to present a direct and much simpler proof here). Since $(T * \check{\rho_n} * \varphi)(x) = \langle T, \varphi(x - \cdot) * \rho_n \rangle$ and $(T * \varphi)(x) = \langle T, \varphi(x - \cdot) \rangle$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we need to prove that

$$\lim_{n \to +\infty} \Biggl\langle T, \Bigl[\varphi(x - \cdot) * \rho_n - \varphi(x - \cdot) \Bigr] \Biggr\rangle = 0,$$

uniformly for $x \in \mathbb{R}$. In the Beurling case, we have the existence of a positive real number $h > 0$ such that, due to a simple calculation involving the condition (M.2), the mean value theorem and the continuity of T on $\mathcal{D}^{(M_p)}_{[-1,1]}$,

$$\Bigl\|\Bigl\langle T, \Bigl[\varphi(x-\cdot) * \rho_n - \varphi(x-\cdot)\Bigr]\Bigr\rangle\Bigr\|$$
$$\leq \sup_{p\geq 0} \frac{h^p \sup_{y\in[-1,1]}\Bigl|\Bigl[\varphi^{(p)}(x-\cdot) * \rho_n - \varphi^{(p)}(x-\cdot)\Bigr](y)\Bigr|}{M_p}$$
$$\leq \sup_{p\geq 0} \frac{h^p \int_0^1 \Bigl|\varphi^{(p)}(x-y+t) - \varphi^{(p)}(x-y)\Bigr|\rho_n(t)\,dt}{M_p}$$
$$\leq \int_0^1 t\rho_n(t)\,dt \ \cdot\ \sup_{p\geq 0} \frac{h^{p+1}\Bigl\|\varphi^{(p+1)}\Bigr\|_\infty}{M_p}$$
$$\leq \frac{1}{n}\int_0^1 t\rho(t)\,dt \ \cdot\ \frac{1}{AM_1 h}\sup_{p\geq 0} \frac{(h/H)^{p+1}\Bigl\|\varphi^{(p+1)}\Bigr\|_\infty}{M_{p+1}}.$$

In the Roumieu case, the above holds for all positive real numbers $h > 0$, which simply completes the proof of proposition. □

If $f : \mathbb{R} \to X$ is uniformly continuous, $f \in \mathcal{D}'^{*}_{\mathrm{A}}(X)$ and $\mathrm{A} \cap C(\mathbb{R} : X)$ is closed under uniform convergence, then we can use the fact that $\lim_{n\to\infty} \rho_n * f = f$, uniformly on $\mathbb{R}$, to get that $f \in \mathrm{A}$.

The statements of [4, Lemma 2.3, Proposition 2.4] hold in ultradistribution case, so that the assumption $\triangle_h T := T_{-h} - T \in \mathcal{D}'^{*}_{\mathbb{A}}(X)$ for all $h > 0$ implies $T - \widetilde{M_h} T \in \mathcal{D}'^{*}_{\mathbb{A}}(X)$, as well as:

- (i) $\mathcal{D}'^{*}_{\mathrm{A}}(X) \subseteq \widetilde{\mathcal{M}}\mathcal{D}'^{*}_{\mathrm{A}}(X) = \mathcal{D}'^{*}_{\widetilde{\mathcal{M}}\mathrm{A}}(X)$, and
- (ii) $\mathcal{D}'^{*}_{\mathrm{A}}(X) \subseteq \widetilde{\mathcal{M}}\mathcal{D}'^{*}_{\mathrm{A}}(X)$ if A is a cone (i.e., $[0,\infty)\cdot \mathrm{A} + [0,\infty)\cdot \mathrm{A} \subseteq \mathrm{A}$) satisfying $(\triangle_1)$.

It is worth noting that [4, Corollary 2.5, Corollary 2.6], results from [4, Section 4] and [3, Proposition 1.1] can be formulated in ultradistribution case, as well.

The following statements are in a close connection with [11, Theorem 2] and [4, Theorem 2.11]:

1. Let there exist an ultradifferential operator $P(D) = \sum_{p=0}^{\infty} a_p D^p$ of class (M_p), resp., of class $\{M_p\}$, and $f,\ g \in \mathcal{D}'^{*}_{\mathrm{A}}(X)$ such that $T = P(D)f + g$. If A is closed under addition, then $T \in \mathcal{D}'^{*}_{\mathrm{A}}(X)$.

2. If $\mathrm{A}\cap C(\mathbb{R}:X)$ is closed under uniform convergence, $T\in\mathcal{D}'_{L^1}((M_p):X)$ and $T*\varphi\in\mathrm{A}$, $\varphi\in\mathcal{D}^{(M_p)}$, then there exists $h>0$ such that for each compact set $K\subseteq\mathbb{R}$ we have $T*\varphi\in\mathrm{A}$, $\varphi\in\mathcal{D}_K^{M_p,h}$. In our personal opinion, we need to assume here that $T\in\mathcal{D}'_{L^1}((M_p):X)$ since the set-theoretical equality appearing on the second line of proof of implication [11, (iii) $\Rightarrow$ (iv)] is mistakenly written.
3. Assume $T\in\mathcal{D}'^{(M_p)}(X)$ and there exists $h>0$ such that for each compact set $K\subseteq\mathbb{R}$ we have $T*\varphi\in\mathrm{A}$, $\varphi\in\mathcal{D}_K^{M_p,h}$. If (M_p) additionally satisfies (M.3), then there exist $l>0$ and two elements $f,\ g\in\mathrm{A}$ such that $T=P(D)f+g$.

4. Conclusions and final remarks

In this section, we will present several conclusions and remarks about the obtained results, propose some open problems and possible ways for expanding this research. First of all, we would like to raise the following issues that we have not been able to solve:

A. Theorem 1: Let (M_p) satisfy the conditions (M.1), (M.2) and (M.3'), and let $T\in\mathcal{D}'^*(X)$. Is it true that the assertion (iv) implies (ii)?
B. Theorem 2 and Remark 2: Let (M_p) satisfy the conditions (M.1), (M.2) and (M.3'), and let $T\in\mathcal{D}'_{L^1}((M_p):X)$, resp. $T\in\mathcal{D}'_{L^1}(\{M_p\}:X)$. Is it true that (ii)' is equivalent with (ii), in orther words, is it true that (ii) implies (ii)'?

The notion of space $\mathcal{D}'^*_{\mathrm{A}}(X)$ can be modified for tempered vector-valued ultradistributions as follows

$$\mathcal{S}'^*_{\mathrm{A}}(X):=\Biggl\{T\in\ \mathcal{S}'^*(X):T*\varphi\in\mathrm{A}\text{ for all }\varphi\in\mathcal{S}^*\Biggr\}.$$

Concerning some known results on the structure of space $\mathcal{S}'^*_{\mathrm{A}}(X)$, we would like to note that P. Dimovski, B. Prangoski and D. Velinov [13] have examined the convolutors and the space of multipliers of Beurling and Roumieu tempered scalar-valued ultradistributions. We feel duty bound to observe that [13, Proposition 3.2] can be formulated in the vector-valued case, as well, and that some implications from the formulation of this result holds even in the case that the sequence (M_p) does not satisfy (M.3). To precise this, we introduce the space of vector-valued convolutors $O'_C(X)$ of $\mathcal{S}'^*(X)$ as the space consisted of all tempered vector-valued ultradistributions $T\in\mathcal{S}'^*(X)$ such that, for every $\varphi\in\mathcal{S}^*$, we have $T*\varphi\in\mathcal{S}^*(X)$ and that the mapping $\varphi\mapsto T*\varphi$, $\mathcal{S}^*\to\mathcal{S}^*(X)$ is continuous. We have the following result:

Proposition 2. *Suppose that* (M_p) *satisfies the conditions* (M.1), (M.2) *and* (M.3'). *Let* $T\in\mathcal{S}'^*(X)$. *Then we have* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv), *where:*

(i) $T\in O'_C(X)$.

(ii) *For every* $\varphi \in \mathcal{S}^*$, *we have* $T * \varphi \in \mathcal{S}^*(X)$, *i.e.,* $T \in \mathcal{S}'^*_{\mathcal{S}^*(X)}(X)$.

(iii) *For every* $\varphi \in \mathcal{D}^*$, *we have* $T * \varphi \in \mathcal{S}^*(X)$.

(iv) *For every* $r > 0$, *resp., there exists* $r > 0$, *such that the set*

$$\left\{ e^{M(r|h|)} T_h : h \in \mathbb{R} \right\}$$

is bounded in $\mathcal{D}'^*(X)$.

Moreover, if (M_p) *satisfies* (M.3), *then we have* (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v), *where:*

(v) *For every* $r > 0$, *resp., there exists* $r > 0$, *there exists* $l > 0$, *resp., there exists a sequence* (r_p) *of positive real numbers tending to infinity, and two functions* $f,\ g \in L^\infty(\mathbb{R} : X)$ *such that* $T = P_l(D)f + g$, *resp.,* $T = T_{r_p}(D)f + g$, *and*

$$\sup_{x \in \mathbb{R}} \left\| e^{M(x|h|)} \left[\|f(x)]\| + \|g(x)\| \right] \right\| < \infty.$$

It is almost impossible to summarize here all obtained results about the structure of spaces $\mathcal{D}'^*_{\mathrm{A}}(X)$ and $\mathcal{S}'^*_{\mathrm{A}}(X)$ in general case. We would like to mention here only one more result in this direction, obtained recently by P. Dimovski, S. Pilipović, B. Prangoski and J. Vindas [14]. They have introduced the notion of a translation-invariant Banach space of tempered ultradistributions of $*$-class and structurally characterized the space $\mathcal{D}'^*_{E'}(\mathbb{C})$, where E is a translation-invariant Banach space (see [14, Definition 4.1, Theorem 6.1]). It would be very interesting to prove a vector-valued version of this result, as well as vector-valued versions of [9, Theorem 2.3.1, Theorem 2.3.2] and some results from [15] and [25].

In [4, Theorem 2.15], B. Basit and H. Güenzler have proved several equivalent conditions for a vector-valued distribution $T \in \mathcal{D}'(X)$ to belong the space $\mathcal{D}'_{\mathcal{S}'(X)}(X)$, i.e., that $T * \varphi \in \mathcal{S}'(X)$, $\varphi \in \mathcal{D}$. As an interesting problem for our readers, we would like to address the problem of structural characterization of vector-valued ultradistributions $T \in \mathcal{D}'^*(X)$ for which $T * \varphi \in \mathcal{S}'^*(X)$, $\varphi \in \mathcal{D}^*$. This problem seems to be unsolved even in scalar-valued case.

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