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INDEPENDENCE OF CHARACTERIZING PROPERTIES OF $(m+k,m)-$RECTANGULAR BANDS

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Abstract. An $(m+k,m)-$semigroup $(Q;[\ ])$ which is a direct product of a left-zero $(m+k,m)-$semigroup and a right-zero $(m+k,m)-$semigroup is called an $(m+k,m)-$rectangular band. The class of $(m+k,m)-$rectangular bands is characterized by the following identities:
(I) $\left[x_1^{m+k}\right]_i = \left[y_1^{j-1}x_iy_{j+1}^{j+k-1}x_{i+k}y_{j+k+1}^{m+k}\right]_j$, $i,j\in\mathbb{N}_m$;
(II) $\left[x_1^{m+2k}\right]_i = \left[x_1^i x_{i+k+1}^{m+2k}\right]_i$;
(III) $\left[\overset{m+k}{x}\right] = \overset{m}{x}$.
Here, the independence between these identities is proved.

1. Preliminaries

For a set Q and a positive integer s, Q^s denotes the s-th Cartesian power of Q. We use the notation x_1^s for the elements of Q^s. If $x_1 = x_2 = \cdots = x_s = x$, then x_1^s is denoted by the symbol $\overset{s}{x}$.

Let n,m be positive integers. An $(n,m)-$groupoid is a pair $(Q;[\ ])$ where $Q\neq\emptyset$ and $[\ ]$ is an $(n,m)-$operation i.e. a map $[\ ]: Q^n\to Q^m$. Every $(n,m)-$operation on Q induces a sequence $[\ ]_1,[\ ]_2,\ldots,[\ ]_m$ of $n-$ary operations on the set Q, such that

$$((\forall i\in\mathbb{N}_m)\quad [x_1^n]_i = y_i)\Leftrightarrow [x_1^n] = y_1^m.$$

Let $m\geq 2$, $k\geq 1$. An $(m+k,m)-$groupoid $(Q;[\ ])$ is called an $(m+k,m)-$semigroup if for each $i\in\{0,1,2,\ldots,k\}$

$$\left[x_1^i\left[x_{i+1}^{i+m+k}\right]x_{i+m+k+1}^{m+2k}\right] = \left[\left[x_1^{m+k}\right]x_{m+k+1}^{m+2k}\right].$$

Let $\mathbf{A}=(A,[\ ])$ be an $(m+k,m)-$groupoid, where the $(m+k,m)-$operation $[\ ]$ is defined by $[x_1^{m+k}] = x_1^m$. Then $\mathbf{A}$ is an $(m+k,m)-$semigroup and it is called a left-zero $(m+k,m)-$semigroup.

Dually, a right-zero $(m+k,m)-$semigroup $\mathbf{B}=(B,[\ ])$ is defined by the operation $[x_1^{m+k}] = x_{k+1}^{m+k}$.

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The pair $(A \times B; [\,])$, where $[\,]$ is an $(m+k,m)-$operation on $A \times B$ defined by

$$\left[x_1^{m+k}\right] = y_1^m \Leftrightarrow (x_i = (a_i, b_i), y_j = (a_j, b_{j+k}), i \in \mathbb{N}_{m+k}, j \in \mathbb{N}_m)$$

is an $(m+k,m)-$semigroup and it is a direct product of a left-zero (**A**) and a right-zero (**B**) $(m+k,m)-$semigroup. Such an $(m+k,m)-$semigroup is called $(m+k,m)-$rectangular band.

2. Characterization of $(m+k,m)-$rectangular bands

A characterization of $(2k,k)-$rectangular band is given in [3] and a characterization of $(m+k,m)-$rectangular band when $k < m$ is given in [4]. Here we will give a characterization of $(m+k,m)-$rectangular band in general.

Theorem 2.1. *An $(m+k,m)-$semigroup $\mathbf{Q} = (Q, [\,])$ is an $(m+k,m)-$rectangular band if and only if the following conditions are satisfied in* $\mathbf{Q}$:

(I) $\left[x_1^{m+k}\right]_i = \left[y_1^{j-1} x_i y_{j+1}^{j+k-1} x_{i+k} y_{j+k+1}^{m+k}\right]_j$, $i, j \in \mathbb{N}_m$;

(II) $\left[x_1^{m+2k}\right]_i = \left[x_1^i x_{i+k+1}^{m+2k}\right]_i$;

(III) $\left[\overset{m+k}{x}\right] = \overset{m}{x}$.

Proof. Suppose that the $(m+k,m)-$semigroup satisfies (I), (II) and (III), and a is a fixed element of Q.

(A) Denote by L the subset of Q, $L = \left\{ \left[x \overset{m+k-1}{a}\right]_1 \mid x \in Q \right\}$.

Let $\left[x_i \overset{m+k-1}{a}\right]_1 \in L$, $i \in \mathbb{N}_{m+k}$. Then:

$$\left[\left[x_1 \overset{m+k-1}{a}\right]_1 \ldots \left[x_{m+k} \overset{m+k-1}{a}\right]_1\right]_i \overset{\text{I}}{=} \left[\left[x_i \overset{m+k-1}{a}\right]_1 \overset{k-1}{a} \left[x_{i+k} \overset{m+k-1}{a}\right]_1 \overset{m-1}{a}\right]_1$$

$$\overset{\text{I}}{=} \left[\left[x_i \overset{m+k-1}{a}\right]_1 \overset{k-1}{a} \left[x_{i+k} \overset{m+k-1}{a}\right]_1 \left[x_{i+k} \overset{m+k-1}{a}\right]_2 \ldots \left[x_{i+k} \overset{m+k-1}{a}\right]_m\right]_1$$

$$= \left[\left[x_i \overset{m+k-1}{a}\right]_1 \overset{k-1}{a} x_{i+k} \overset{m+k-1}{a}\right]_1 \overset{\text{II}}{=} \left[\left[x_i \overset{m+k-1}{a}\right]_1 \overset{m+k-1}{a}\right]_1$$

$$\overset{\text{I}}{=} \left[\overset{m-1}{a} \left[\overset{m-1}{a} x_i \overset{k-1}{a} a\right]_m \overset{k-1}{a} a\right]_m$$

$$\overset{\text{I}}{=} \left[\left[\overset{m-1}{a} x_i \overset{k-1}{a} a\right]_1 \ldots \left[\overset{m-1}{a} x_i \overset{k-1}{a} a\right]_{m-1} \left[\overset{m-1}{a} x_i \overset{k-1}{a} a\right]_m \overset{k-1}{a} a\right]_m$$

$$= \left[\overset{m-1}{a} x_i \overset{k-1}{a} a \overset{k-1}{a} a\right]_m \overset{\text{II}}{=} \left[\overset{m-1}{a} x_i \overset{k-1}{a} a\right]_m \overset{\text{I}}{=} \left[x_i \overset{k-1}{a} a \overset{m-1}{a}\right]_1 = \left[x_i \overset{m+k-1}{a}\right]_1.$$

So, $(L, [\,])$ is a left-zero $(m+k,m)-$semigroup.

(B) Let $D = \left\{ \left[\overset{m+k-1}{a} x\right]_m \mid x \in Q \right\}$ and $x_i \in Q, i \in \mathbb{N}_{m+k}$. Then:

$$\left[\left[\overset{m+k-1}{a} x_1\right]_m \ldots \left[\overset{m+k-1}{a} x_{m+k}\right]_m\right]_i \overset{\text{I}}{=} \left[\overset{m-1}{a} \left[\overset{m+k-1}{a} x_i\right]_m \overset{k-1}{a} \left[\overset{m+k-1}{a} x_{i+k}\right]_m\right]_m$$

$$\stackrel{\mathrm{I}}{=} \left[\left[\overset{m+k-1}{a} \; x_i \right]_1 \ldots \left[\overset{m+k-1}{a} \; x_i \right]_{m-1} \left[\overset{m+k-1}{a} \; x_i \right]_m \overset{k-1}{a} \left[\overset{m+k-1}{a} \; x_{i+k} \right]_m \right]_m$$

$$= \left[\overset{m+k-1}{a} \; x_i \overset{k-1}{a} \left[\overset{m+k-1}{a} \; x_{i+k} \right]_m \right]_m \stackrel{\mathrm{II}}{=} \left[\overset{m}{a} \overset{k-1}{a} \left[\overset{m+k-1}{a} \; x_{i+k} \right]_m \right]_m$$

$$\stackrel{\mathrm{I}}{=} \left[a \overset{k-1}{a} \left[a \overset{k-1}{a} x_{i+k} \overset{m-1}{a} \right]_1 \overset{m-1}{a} \right]_1$$

$$\stackrel{\mathrm{I}}{=} \left[a \overset{k-1}{a} \left[a \overset{k-1}{a} x_{i+k} \overset{m-1}{a} \right]_1 \left[a \overset{k-1}{a} x_{i+k} \overset{m-1}{a} \right]_2 \ldots \left[a \overset{k-1}{a} x_{i+k} \overset{m-1}{a} \right]_m \right]_1$$

$$= \left[a \overset{k-1}{a} a \overset{k-1}{a} x_{i+k} \overset{m-1}{a} \right]_1 \stackrel{\mathrm{II}}{=} \left[a \overset{k-1}{a} x_{i+k} \overset{m-1}{a} \right]_1 \stackrel{\mathrm{I}}{=} \left[\overset{m-1}{a} a \overset{k-1}{a} x_{i+k} \right]_m = \left[\overset{m+k-1}{a} \; x_{i+k} \right]_m .$$

So, $(D, [\;])$ is a right-zero $(m+k, m)-$semigroup.

(C) We define a map $\varphi : L \times D \to Q$ with:

$$\varphi\left(\left[x \overset{m+k-1}{a} \right]_1 , \left[\overset{m+k-1}{a} \; y \right]_m \right) = \left[x \overset{k-1}{a} y \overset{m-1}{a} \right]_1 ,$$

for all $\left(\left[x \overset{m+k-1}{a} \right]_1 , \left[\overset{m+k-1}{a} \; y \right]_m \right) \in L \times D.$

(C1) We will prove that φ is a well-defined map.

Let $\left[x \overset{m+k-1}{a} \right]_1 = \left[u \overset{m+k-1}{a} \right]_1 , \left[\overset{m+k-1}{a} \; y \right]_m = \left[\overset{m+k-1}{a} \; v \right]_m$. Then:

(C1.1) $\left[x \overset{m+k-1}{a} \right]_1 = \left[u \overset{m+k-1}{a} \right]_1$

$$\left[\overset{m-1}{a} \left[x \overset{m+k-1}{a} \right]_1 \overset{k-1}{a} x \right]_m = \left[\overset{m-1}{a} \left[u \overset{m+k-1}{a} \right]_1 \overset{k-1}{a} x \right]_m$$

$$\left[\overset{m-1}{a} \left[\overset{m-1}{a} x \overset{k-1}{a} a \right]_m \overset{k-1}{a} x \right]_m = \left[\overset{m-1}{a} \left[\overset{m-1}{a} u \overset{k-1}{a} a \right]_m \overset{k-1}{a} x \right]_m$$

$$\left[\left[\overset{m-1}{a} x \overset{k-1}{a} a \right]_1 \ldots \left[\overset{m-1}{a} x \overset{k-1}{a} a \right]_{m-1} \left[\overset{m-1}{a} x \overset{k-1}{a} a \right]_m \overset{k-1}{a} x \right]_m$$

$$= \left[\left[\overset{m-1}{a} u \overset{k-1}{a} a \right]_1 \ldots \left[\overset{m-1}{a} u \overset{k-1}{a} a \right]_{m-1} \left[\overset{m-1}{a} u \overset{k-1}{a} a \right]_m \overset{k-1}{a} x \right]_m$$

$$\left[\overset{m-1}{a} x \overset{k-1}{a} a \overset{k-1}{a} x \right]_m = \left[\overset{m-1}{a} u \overset{k-1}{a} a \overset{k-1}{a} x \right]_m$$

$$\left[\overset{m-1}{a} x \overset{k-1}{a} x \right]_m = \left[\overset{m-1}{a} u \overset{k-1}{a} x \right]_m$$

$$\left[\overset{m+k}{x} \right]_m = \left[\overset{m-1}{a} u \overset{k-1}{a} x \right]_m .$$

So, $x = \left[\overset{m-1}{a}\ u\ \overset{k-1}{a}\ x\right]_m$.

(C1.2) $\left[\overset{m+k-1}{a}\ y\right]_m = \left[\overset{m+k-1}{a}\ v\right]_m$

$$\left[y\ \overset{k-1}{a}\left[\overset{m+k-1}{a}\ y\right]_m \overset{m-1}{a}\right]_1 = \left[y\ \overset{k-1}{a}\left[\overset{m+k-1}{a}\ v\right]_m \overset{m-1}{a}\right]_1$$

$$\left[y\ \overset{k-1}{a}\left[a\ \overset{k-1}{a}\ y\ \overset{m-1}{a}\right]_1 \overset{m-1}{a}\right]_1 = \left[y\ \overset{k-1}{a}\left[a\ \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_1 \overset{m-1}{a}\right]_1$$

$$\left[y\ \overset{k-1}{a}\left[a\ \overset{k-1}{a}\ y\ \overset{m-1}{a}\right]_1\left[a\ \overset{k-1}{a}\ y\ \overset{m-1}{a}\right]_2 \dots \left[a\ \overset{k-1}{a}\ y\ \overset{m-1}{a}\right]_m\right]_1$$

$$= \left[y\ \overset{k-1}{a}\left[a\ \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_1\left[a\ \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_2 \dots \left[a\ \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_m\right]_1$$

$$\left[y\ \overset{k-1}{a}\ a\ \overset{k-1}{a}\ y\ \overset{m-1}{a}\right]_1 = \left[y\ \overset{k-1}{a}\ a\ \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_1$$

$$\left[y\ \overset{k-1}{a}\ y\ \overset{m-1}{a}\right]_1 = \left[y\ \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_1$$

$$\left[\overset{m+k}{y}\right]_1 = \left[y\ \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_1 .$$

So, $y = \left[y\ \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_1$.

Then:

$\left[x\ \overset{k-1}{a}\ y\ \overset{m-1}{a}\right]_1 = \left[\left[\overset{m-1}{a}\ u\ \overset{k-1}{a}\ x\right]_m \overset{k-1}{a}\left[y\ \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_1 \overset{m-1}{a}\right]_1$

$\overset{\mathrm{I}}{=} \left[\left[\overset{m-1}{a}\ u\ \overset{k-1}{a}\ x\right]_m \overset{k-1}{a}\left[y\ \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_1\left[y\ \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_2 \dots \left[y\ \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_m\right]_1$

$= \left[\left[\overset{m-1}{a}\ u\ \overset{k-1}{a}\ x\right]_m \overset{k-1}{a}\ y\ \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_1 \overset{\mathrm{II}}{=} \left[\left[\overset{m-1}{a}\ u\ \overset{k-1}{a}\ x\right]_m \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_1$

$\overset{\mathrm{I}}{=} \left[\overset{m-1}{a}\left[\overset{m-1}{a}\ u\ \overset{k-1}{a}\ x\right]_m \overset{k-1}{a}\ v\right]_m$

$\overset{\mathrm{I}}{=} \left[\left[\overset{m-1}{a}\ u\ \overset{k-1}{a}\ x\right]_1 \dots \left[\overset{m-1}{a}\ u\ \overset{k-1}{a}\ x\right]_{m-1}\left[\overset{m-1}{a}\ u\ \overset{k-1}{a}\ x\right]_m \overset{k-1}{a}\ v\right]_m = \left[\overset{m-1}{a}\ u\ \overset{k-1}{a}\ x\ \overset{k-1}{a}\ v\right]_m$

$\overset{\mathrm{II}}{=} \left[\overset{m-1}{a}\ u\ \overset{k-1}{a}\ v\right]_m \overset{\mathrm{I}}{=} \left[u\ \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_1$.

Therefore, φ is a well-defined map.

(C2) We will prove that φ is an injection.

Let $\varphi\left(\left[x\ \overset{m+k-1}{a}\right]_1, \left[\overset{m+k-1}{a}\ y\right]_m\right) = \varphi\left(\left[u\ \overset{m+k-1}{a}\right]_1, \left[\overset{m+k-1}{a}\ v\right]_m\right)$,

i.e. $\left[x\ \overset{k-1}{a}\ y\ \overset{m-1}{a}\right]_1 = \left[u\ \overset{k-1}{a}\ v\ \overset{m-1}{a}\right]_1$. Then:

(C2.1) $\left[x\overset{k-1}{a}y\overset{m-1}{a}\right]_1=\left[u\overset{k-1}{a}v\overset{m-1}{a}\right]_1$

$$\left[\overset{m-1}{a}\left[x\overset{k-1}{a}y\overset{m-1}{a}\right]_1\overset{k-1}{a}x\right]_m=\left[\overset{m-1}{a}\left[u\overset{k-1}{a}v\overset{m-1}{a}\right]_1\overset{k-1}{a}x\right]_m$$

$$\left[\overset{m-1}{a}\left[\overset{m-1}{a}x\overset{k-1}{a}y\right]_m\overset{k-1}{a}x\right]_m=\left[\overset{m-1}{a}\left[\overset{m-1}{a}u\overset{k-1}{a}v\right]_m\overset{k-1}{a}x\right]_m$$

$$\left[\left[\overset{m-1}{a}x\overset{k-1}{a}y\right]_1\ldots\left[\overset{m-1}{a}x\overset{k-1}{a}y\right]_{m-1}\left[\overset{m-1}{a}x\overset{k-1}{a}y\right]_m\overset{k-1}{a}x\right]_m$$

$$=\left[\left[\overset{m-1}{a}u\overset{k-1}{a}v\right]_1\ldots\left[\overset{m-1}{a}u\overset{k-1}{a}v\right]_{m-1}\left[\overset{m-1}{a}u\overset{k-1}{a}v\right]_m\overset{k-1}{a}x\right]_m$$

$$\left[\overset{m-1}{a}x\overset{k-1}{a}y\overset{k-1}{a}x\right]_m=\left[\overset{m-1}{a}u\overset{k-1}{a}v\overset{k-1}{a}x\right]_m$$

$$\left[\overset{m-1}{a}x\overset{k-1}{a}x\right]_m=\left[\overset{m-1}{a}u\overset{k-1}{a}x\right]_m$$

$$\left[\overset{m+k}{x}\right]_m=\left[\overset{m-1}{a}u\overset{k-1}{a}x\right]_m.$$

So, $x=\left[\overset{m-1}{a}u\overset{k-1}{a}x\right]_m$.

We have:

$$\left[x\overset{m+k-1}{a}\right]_1=\left[\left[\overset{m-1}{a}u\overset{k-1}{a}x\right]_m\overset{m+k-1}{a}\right]_1\overset{\mathrm{I}}{=}\left[\overset{m-1}{a}\left[\overset{m-1}{a}u\overset{k-1}{a}x\right]_m\overset{k-1}{a}a\right]_m$$

$$\overset{\mathrm{I}}{=}\left[\left[\overset{m-1}{a}u\overset{k-1}{a}x\right]_1\ldots\left[\overset{m-1}{a}u\overset{k-1}{a}x\right]_{m-1}\left[\overset{m-1}{a}u\overset{k-1}{a}x\right]_m\overset{k-1}{a}a\right]_m=\left[\overset{m-1}{a}u\overset{k-1}{a}x\overset{k-1}{a}a\right]_m$$

$$\overset{\mathrm{II}}{=}\left[\overset{m-1}{a}u\overset{k-1}{a}a\right]_m\overset{\mathrm{I}}{=}\left[u\overset{k-1}{a}a\overset{m-1}{a}\right]_1=\left[u\overset{m+k-1}{a}\right]_1.$$

(C2.2) $\left[x\overset{k-1}{a}y\overset{m-1}{a}\right]_1=\left[u\overset{k-1}{a}v\overset{m-1}{a}\right]_1$

$$\left[y\overset{k-1}{a}\left[x\overset{k-1}{a}y\overset{m-1}{a}\right]_1\overset{m-1}{a}\right]_1=\left[y\overset{k-1}{a}\left[u\overset{k-1}{a}v\overset{m-1}{a}\right]_1\overset{m-1}{a}\right]_1$$

$$\left[y\overset{k-1}{a}\left[x\overset{k-1}{a}y\overset{m-1}{a}\right]_1\left[x\overset{k-1}{a}y\overset{m-1}{a}\right]_2\ldots\left[x\overset{k-1}{a}y\overset{m-1}{a}\right]_m\right]_1$$

$$=\left[y\overset{k-1}{a}\left[u\overset{k-1}{a}v\overset{m-1}{a}\right]_1\left[u\overset{k-1}{a}v\overset{m-1}{a}\right]_2\ldots\left[u\overset{k-1}{a}v\overset{m-1}{a}\right]_m\right]_1$$

$$\left[y\overset{k-1}{a}x\overset{k-1}{a}y\overset{m-1}{a}\right]_1=\left[y\overset{k-1}{a}u\overset{k-1}{a}v\overset{m-1}{a}\right]_1$$

$$\left[y\overset{k-1}{a}y\overset{m-1}{a}\right]_1=\left[y\overset{k-1}{a}v\overset{m-1}{a}\right]_1$$

$$\left[\overset{m+k}{y}\right]_1 = \left[y \overset{k-1}{a} v \overset{m-1}{a}\right]_1$$
$$y = \left[y \overset{k-1}{a} v \overset{m-1}{a}\right]_1 .$$

Then:

$$\left[\overset{m+k-1}{a} y\right]_m = \left[\overset{m+k-1}{a} \left[y \overset{k-1}{a} v \overset{m-1}{a}\right]_1\right]_m \overset{\mathrm{I}}{=} \left[a \overset{k-1}{a} \left[y \overset{k-1}{a} v \overset{m-1}{a}\right]_1 \overset{m-1}{a}\right]_1$$
$$\overset{\mathrm{I}}{=} \left[a \overset{k-1}{a} \left[y \overset{k-1}{a} v \overset{m-1}{a}\right]_1 \left[y \overset{k-1}{a} v \overset{m-1}{a}\right]_2 \dots \left[y \overset{k-1}{a} v \overset{m-1}{a}\right]_m\right]_1 = \left[a \overset{k-1}{a} y \overset{k-1}{a} v \overset{m-1}{a}\right]_1$$
$$\overset{\mathrm{II}}{=} \left[a \overset{k-1}{a} v \overset{m-1}{a}\right]_1 \overset{\mathrm{I}}{=} \left[\overset{m-1}{a} a \overset{k-1}{a} v\right]_m = \left[\overset{m+k-1}{a} v\right]_m .$$

So, $\left(\left[x \overset{m+k-1}{a}\right]_1, \left[\overset{m+k-1}{a} y\right]_m\right) = \left(\left[u \overset{m+k-1}{a}\right]_1, \left[\overset{m+k-1}{a} v\right]_m\right)$, i.e. φ is an injection.

(C3) We will prove that φ is an surjection.
Let $x \in Q$. Then $\left[x \overset{m+k-1}{a}\right]_1 \in L$, $\left[\overset{m+k-1}{a} x\right]_m \in D$ and

$$\varphi\left(\left[x \overset{m+k-1}{a}\right]_1, \left[\overset{m+k-1}{a} x\right]_m\right) = \left[x \overset{k-1}{a} x \overset{m-1}{a}\right]_1 \overset{\mathrm{I}}{=} \left[\overset{m+k}{x}\right]_1 \overset{\mathrm{III}}{=} x.$$

(C4) We will prove that φ is an $(m+k, m)-$homomorphism.
Let $\alpha_i = \left(\left[x_i \overset{m+k-1}{a}\right]_1, \left[\overset{m+k-1}{a} y_i\right]_m\right) \in L \times D, i \in \mathbb{N}_{m+k}$.
Then:

$$\left[\alpha_1^{m+k}\right]_i$$
$$= \left[\left(\left[x_1 \overset{m+k-1}{a}\right]_1, \left[\overset{m+k-1}{a} y_1\right]_m\right) \dots \left(\left[x_{m+k} \overset{m+k-1}{a}\right]_1, \left[\overset{m+k-1}{a} y_{m+k}\right]_m\right)\right]_i$$
$$= \left(\left[\left[x_1 \overset{m+k-1}{a}\right]_1 \dots \left[x_{m+k} \overset{m+k-1}{a}\right]_1\right]_i, \left[\left[\overset{m+k-1}{a} y_1\right]_m \dots \left[\overset{m+k-1}{a} y_{m+k}\right]_m\right]_i\right)$$
$$= \left(\left[x_i \overset{m+k-1}{a}\right]_1, \left[\overset{m+k-1}{a} y_{i+k}\right]_m\right)$$

and

$$\varphi(\left[\alpha_1^{m+k}\right]_i) = \varphi\left(\left[x_i \overset{m+k-1}{a}\right]_1, \left[\overset{m+k-1}{a} y_{i+k}\right]_m\right) = \left[x_i \overset{k-1}{a} y_{i+k} \overset{m-1}{a}\right]_1 .$$

On the other hand, $\varphi(\alpha_i) = \varphi\left(\left[x_i \overset{m+k-1}{a}\right]_1, \left[\overset{m+k-1}{a} y_i\right]_m\right) = \left[x_i \overset{k-1}{a} y_i \overset{m-1}{a}\right]_1$.

Then:

$$[\varphi(\alpha_1) \dots \varphi(\alpha_{m+k})]_i = \left[\left[x_1 \overset{k-1}{a} y_1 \overset{m-1}{a}\right]_1 \dots \left[x_{m+k} \overset{k-1}{a} y_{m+k} \overset{m-1}{a}\right]_1\right]_i$$

$$\overset{\mathrm{I}}{=} \left[\left[x_i \overset{k-1}{a} y_i \overset{m-1}{a} \right]_1 \overset{k-1}{a} \left[x_{i+k} \overset{k-1}{a} y_{i+k} \overset{m-1}{a} \right]_1 \overset{m-1}{a} \right]_1$$
$$\overset{\mathrm{I}}{=} \left[\left[x_i \overset{k-1}{a} y_i \overset{m-1}{a} \right]_1 \overset{k-1}{a} \left[x_{i+k} \overset{k-1}{a} y_{i+k} \overset{m-1}{a} \right]_1 \dots \left[x_{i+k} \overset{k-1}{a} y_{i+k} \overset{m-1}{a} \right]_m \right]_1$$
$$= \left[\left[x_i \overset{k-1}{a} y_i \overset{m-1}{a} \right]_1 \overset{k-1}{a} x_{i+k} \overset{k-1}{a} y_{i+k} \overset{m-1}{a} \right]_1 \overset{\mathrm{II}}{=} \left[\left[x_i \overset{k-1}{a} y_i \overset{m-1}{a} \right]_1 \overset{k-1}{a} y_{i+k} \overset{m-1}{a} \right]_1$$
$$\overset{\mathrm{I}}{=} \left[\overset{m-1}{a} \left[\overset{m-1}{a} x_i \overset{k-1}{a} y_i \right]_m \overset{k-1}{a} y_{i+k} \right]_m$$
$$\overset{\mathrm{I}}{=} \left[\left[\overset{m-1}{a} x_i \overset{k-1}{a} y_i \right]_1 \dots \left[\overset{m-1}{a} x_i \overset{k-1}{a} y_i \right]_{m-1} \left[\overset{m-1}{a} x_i \overset{k-1}{a} y_i \right]_m \overset{k-1}{a} y_{i+k} \right]_m$$
$$= \left[\overset{m-1}{a} x_i \overset{k-1}{a} y_i \overset{k-1}{a} y_{i+k} \right]_m \overset{\mathrm{II}}{=} \left[\overset{m-1}{a} x_i \overset{k-1}{a} y_{i+k} \right]_m \overset{\mathrm{I}}{=} \left[x_i \overset{k-1}{a} y_{i+k} \overset{m-1}{a} \right]_1 .$$

So, $\varphi(\left[\alpha_1^{m+k}\right]_i) = [\varphi(\alpha_1)\dots\varphi(\alpha_{m+k})]_i$ i.e. φ is an $(m+k, m)-$homomorphism.

Hence, $\mathbf{Q}$ is a direct product of a left-zero and a right-zero $(m+k, m)-$semigroup, i.e. $\mathbf{Q}$ is an $(m+k, m)-$rectangular band.

Conversely, let $\mathbf{Q}$ is an $(m+k, m)-$rectangular band.

(D) Let $(x_\alpha, y_\alpha), (u_\beta, y_\beta) \in Q$, $(x_i, y_i) = (u_j, v_j)$, $(x_{i+k}, y_{i+k}) = (u_{j+k}, v_{j+k})$, $\alpha, \beta \in \mathbb{N}_{m+k}$, $i, j \in \mathbb{N}_m$. Then:
$[(x_1, y_1)\dots(x_{m+k}, y_{m+k})]_i = (x_i, y_{i+k}) = (u_j, v_{j+k}) = [(u_1, v_1)\dots(u_{m+k}, v_{m+k})]_j$.
Hence, $\mathbf{Q}$ satisfies (I).

(E) Let $(x_\alpha, y_\alpha) \in Q$, $\alpha \in \mathbb{N}_{m+2k}$. Then:
$[(x_1, y_1)\dots(x_{m+2k}, y_{m+2k})]_i$
$= [[(x_1, y_1)\dots(x_{m+k}, y_{m+k})]\,(x_{m+k+1}, y_{m+k+1})\dots(x_{m+2k}, y_{m+2k})]_i$
$= [(x_1, y_{1+k})\dots(x_m, y_{m+k})(x_{m+k+1}, y_{m+k+1})\dots(x_{m+2k}, y_{m+2k})]_i$
$= (x_i, y_{i+2k}) = [(x_1, y_1)\dots(x_i, y_i)(x_{i+k+1}, y_{i+k+1})\dots(x_{m+2k}, y_{m+2k})]_i$.
Hence, $\mathbf{Q}$ satisfies (II).

(F) Let $(x, y) \in Q$, $i \in \mathbb{N}_m$. Then $\left[\overset{m+k}{(x, y)}\right]_i = (x, y)$, i.e. $\left[\overset{m+k}{(x, y)}\right] = \overset{m}{(x, y)}$.
Hence, $\mathbf{Q}$ satisfies (III). □

In the sections 3, 4 and 5 we will construct three examples of vector valued semigroups that satisfy exactly two of the identities (I), (II) and (III). This way we prove the independence of the characterizing properties.

3. Independence between (III) and the identities (I) and (II)

Example 3.1. Let $(\{0, 1, 2\}; [\ ])$ be $(3, 2)-$groupoid, where the $(3, 2)-$operation is defined by:

$$[xyz] = \begin{cases} xy, & x \neq 2, y \neq 2 \\ 0y, & x = 2, y \neq 2 \\ x0, & x \neq 2, y = 2 \\ 00, & x = y = 2 \end{cases}$$

Let $x, y, z, a \in \{0, 1, 2\}$. Because

$$[[xyz]a] = \begin{cases} [xya]\,, & x \neq 2, y \neq 2 \\ [0ya]\,, & x = 2, y \neq 2 \\ [x0a]\,, & x \neq 2, y = 2 \\ [00a]\,, & x = y = 2 \end{cases} = \begin{cases} xy, & x \neq 2, y \neq 2 \\ 0y, & x = 2, y \neq 2 \\ x0, & x \neq 2, y = 2 \\ 00, & x = y = 2 \end{cases}$$

$$[x[yza]] = \begin{cases} [xyz]\,, & y \neq 2, z \neq 2 \\ [x0z]\,, & y = 2, z \neq 2 \\ [xy0]\,, & y \neq 2, z = 2 \\ [x00]\,, & y = z = 2 \end{cases} = \begin{cases} xy, & x \neq 2, y \neq 2 \\ 0y, & x = 2, y \neq 2 \\ x0, & x \neq 2, y = 2 \\ 00, & x = y = 2 \end{cases}$$

we obtain that $(\{0, 1, 2\}; [\,])$ is a $(3, 2)-$semigroup.

Here (I) holds since for each $x, y, z, a \in \{0, 1, 2\}$

$[xyz]_1 = \begin{cases} x, & x \neq 2 \\ 0, & x = 2 \end{cases} = [xya]_1$, $[xyz]_2 = \begin{cases} y, & y \neq 2 \\ 0, & y = 2 \end{cases} = [ayz]_2$ and

$[xyz]_1 = \begin{cases} x, & x \neq 2 \\ 0, & x = 2 \end{cases} = [axy]_2$.

For $x, y, z, a \in \{0, 1, 2\}$ we have

$[xyza]_1 = \begin{cases} x, & x \neq 2 \\ 0, & x = 2 \end{cases} = [xza]_1$ and $[xyza]_2 = \begin{cases} y, & y \neq 2 \\ 0, & y = 2 \end{cases} = [xya]_2$.

So, (II) is satisfied in $(\{0, 1, 2\}; [\,])$.

Since $[222] = 00$, (III) is not satisfied in $(\{0, 1, 2\}; [\,])$.

Let assume that $(\{0, 1, 2\}; [\,])$ is a $(3, 2)-$rectangular band. Then it is isomorphic to a direct product of one left-zero $(3, 2)-$semigroup $\mathbf{A} = (A; [\,]^A)$ and one right-zero $(3, 2)-$semigroup $\mathbf{B} = (B; [\,]^B)$. Since $|\{0, 1, 2\}| = |A \times B| = |A| \cdot |B|$, we obtain that either $|A| = 3$, $|B| = 1$ or $|A| = 1, |B| = 3$. Let $A = \{a, b, c\}$ and $B = \{d\}$. Then $[(x, d)(y, d)(z, d)]^{A\times B} = (x, d)(y, d)$ and from here $(\{0, 1, 2\}; [\,])$ is a left-zero $(3, 2)-$semigroup, which is false since $[222] = 00$. Similar, in the second case we obtain that $(\{0, 1, 2\}; [\,])$ is a right-zero $(3, 2)-$semigroup, which is false $([222] = 00)$.

Therefore, identity (III) is independent of identities (I) and (II).

4. Independence between (I) and the identities (II) and (III)

The next example shows independence of identity (I) and the rest of the identities.

Example 4.1. Let $(\{0, 1\}; [\,])$ be $(3, 2)-$groupoid, where the $(3, 2)-$operation is defined by:

$$[xyz] = \begin{cases} 00, & x = 0 \\ 11, & x = 1 \end{cases}.$$

Let $x, y, z, a \in \{0, 1\}$. Then

$$[[xyz]\,a] = \begin{cases} [00a]\,, & x = 0 \\ [11a]\,, & x = 1 \end{cases} = \begin{cases} 00, & x = 0 \\ 11, & x = 1 \end{cases} = [x\,[yza]],$$

here for $(\{0, 1\}; [\,])$ is a $(3, 2)-$semigroup.

Let $x, y, z, a \in \{0,1\}$. Using
$[xyza]_1 = \begin{cases} 0, & x=0 \\ 1, & x=1 \end{cases} = [xza]_1$ and $[xyza]_2 = \begin{cases} 0, & x=0 \\ 1, & x=1 \end{cases} = [xya]_2$
it follows that (II) holds in $(\{0,1\};[\,])$.

Since $[000]=00$ and $[111]=11$, (III) is also satisfied in $(\{0,1\};[\,])$. But identity (I) does not hold in $(\{0,1\};[\,])$, since $[011]_2 = 0 \neq 1 = [111]_2$.

Let assume that $(\{0,1\};[\,])$ is a $(3,2)-$rectangular band. Then it is isomorphic to a direct product of one left-zero $(3,2)-$semigroup $\mathbf{A} = (A;[\,]^A)$ and one right-zero $(3,2)-$semigroup $\mathbf{B} = (B;[\,]^B)$. Since $|\{0,1\}| = |A \times B| = |A| \cdot |B|$, we obtain that either $|A|=2$, $|B|=1$ or $|A|=1, |B|=2$. Let $A=\{a,b\}$ and $B=\{c\}$. Then $[(x,c)(y,c)(z,c)]^{A\times B} = (x,c)(y,c)$, and therefore $(\{0,1\};[\,])$ is a left-zero $(3,2)-$semigroup, which is false since $[011]=00$. Similar, in the second case we obtain that $(\{0,1\};[\,])$ is a right-zero $(3,2)-$semigroup, which is false ($[011]=00$).

5. Independence between (II) and the identities (III) and (I)

Example 5.1. Let $(\{0,1\};[\,])$ be $(3,2)-$groupoid, where the $(3,2)-$operation is defined by:

$[010]=[011]=[101]=[110]=[111]=11$; $[000]=00$; $[001]=01$ and $[100]=10$.

Since

$[[000]\,0]=[000]=00=[000]=[0\,[000]]$
$[[000]\,1]=[001]=01=[001]=[0\,[001]]$
$[[001]\,0]=[010]=11=[011]=[0\,[010]]$
$[[001]\,1]=[011]=11=[011]=[0\,[011]]$
$[[010]\,0]=[110]=11=[010]=[0\,[100]]$
$[[010]\,1]=[111]=11=[011]=[0\,[101]]$
$[[011]\,0]=[110]=11=[011]=[0\,[110]]$
$[[011]\,1]=[111]=11=[011]=[0\,[111]]$
$[[100]\,0]=[100]=10=[100]=[1\,[000]]$
$[[100]\,1]=[101]=11=[101]=[1\,[001]]$
$[[101]\,0]=[110]=11=[111]=[1\,[010]]$
$[[101]\,1]=[111]=11=[111]=[1\,[011]]$
$[[110]\,0]=[110]=11=[110]=[1\,[100]]$
$[[110]\,1]=[111]=11=[111]=[1\,[101]]$
$[[111]\,0]=[110]=11=[111]=[1\,[110]]$
$[[111]\,1]=[111]=11=[111]=[1\,[111]]$,

it follows that $(\{0,1\};[\,])$ is a $(3,2)-$semigroup.

Here (I) holds since

$[000]_1=[001]_1=[000]_2=[100]_2=0$
$[010]_1=[011]_1=[001]_2=[101]_2=1$
$[100]_1=[101]_1=[010]_2=[110]_2=1$
$[110]_1=[111]_1=[011]_2=[111]_2=1$.

Also, the identity (III) holds because $[000]=00$ and $[111]=11$, but (II) is not true because $[0100]_1 = 1 \neq 0 = [000]_1$.

The $(3,2)-$semigroup $(\{0,1\};[\,])$ is not $(3,2)-$rectangular band, because if it is $(3,2)-$rectangular band then it is isomorphic to a direct product of one left-zero $(3,2)-$semigroup $\mathbf{A} = (A;[\,]^A)$ and one right-zero $(3,2)-$semigroup $\mathbf{B} = (B;[\,]^B)$, where either $|A| = 2$, $|B| = 1$ or $|A| = 1, |B| = 2$. In that case either $(\{0,1\};[\,]) \cong (A;[\,]^A)$ or $(\{0,1\};[\,]) \cong (B;[\,]^B)$, which is false, $(\{0,1\};[\,])$ is not a left-zero $(3,2)-$semigroup $([001] = 01)$ and right-zero $(3,2)-$semigroup $([110] = 11)$.

Therefore identity (II) is independent of identities (III) and (I).

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НЕЗАВИСНОСТ НА КАРАКТЕРИЗИРАЧКИТЕ СВОЈСТВА НА $(m+k,m)$-ПРАВОАГОЛНИ ЛЕНТИ

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Р е з и м е

$(m+k,m)$-полугрупата $(Q;[\,])$ која е директен производ на една лево-нулта $(m+k,m)-$полугрупа и една десно-нулта $(m+k,m)-$полугрупа се нарекува $(m+k,m)-$правоаголна лента. Класата на $(m+k,m)-$правоаголни ленти се карактеризира со следниве идентитети:

(I) $\left[x_1^{m+k}\right]_i = \left[y_1^{j-1} x_i y_{j+1}^{j+k-1} x_{i+k} y_{j+k+1}^{m+k}\right]_j$, $i,\, j \in \mathbb{N}_m$;

(II) $\left[x_1^{m+2k}\right]_i = \left[x_1^i x_{i+k+1}^{m+2k}\right]_i$;

(III) $\left[\overset{m+k}{x}\right] = \overset{m}{x}$.

Во овој труд, е докажана независноста на идентитетите.

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