Математички Билтен ISSN 0351–336X
30 (LVI)
2006 (37–42)
Скопје, Македонија

A CLASSROOM NOTE

HARMONIC CONJUGATES IN ARBITRARY PLANAR DOMAINS

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Abstract

Eines der ersten Ergebnisse einer einfúhrenden Vorlesung in die Funktionentheorie ist der Satz, dass eine in einem ebenen Gebiet Ω stetige, komplexwertige Funktion f eine holomorphe Stammfuntion besitzt, wenn das komplexe Kurvenintegral $\int_\Gamma f(z)\,dz$ úber jeden geschlossenen Integrationsweg $\Gamma \subseteq \Omega$ werschwindet. Eng damit zuzammenhángend ist die Frage, wann eine harmonische Funktion u in Ω eine harmonisch Konjugierte besitzt; d.h. Wann es eine Funktion v gibt mit $u + iv$ holomorph in Ω. Diese Note soll nun ein erstaunlicherweise nicht in den gángigen Lehrbúchern angegebenes formal áhnliches Kriterium zur Beantwortung dieser Frage herleiten: man ersetze f durch $\frac{\partial u}{\partial n}$, die Normalableitung von u lángs Γ und dz durch des Bogendifferenzial $d\sigma$.

It is well known that every harmonic function, u, in a simply connected planar domain Ω admits a harmonic conjugate v; that is a harmonic function v vor which $f\!:= u + iv$ is holomorphic in Ω. This case of the punctured disk $\{z \in \mathbf{C}: 0 < |z| < 1\}$ and the function $u(z) = \log|z|$ show that the assertion above is no longer true in arbitrary domains. The aim of this elementary note is to present a necessary and sufficient condition for a harmonic function defined on an arbitrary planar domain Ω in order to have a harmonic conjugate. The result, I did not find in any textbook, does not seem to be widely known. Its proof is well suited for presentation in any introductory course on complex function theory.

To begin with, let Ω_m be a finitely connected domain in $\mathbf{C}$ bounded by m pairwise disjoint smooth Jordan curves and let u and v be two functions continuously differentiable on a neighborhood of the closure $\overline{\Omega}_m$ of Ω_m. Then, by taking a suitable orientation of the boundary curves of Ω_m, it follows from Green's theorem ([Ga], p.390) that

$$\int_{\Omega_m} (u\,\Delta v - v\,\Delta u)\,dx\,dy = \int_{\partial\Omega_m} \left(u\,\frac{\partial u}{\partial n} - v\,\frac{\partial u}{\partial n}\right) d\sigma\,, \tag{1}$$

where $d\sigma$ denoted the differentiel with respect to arclenghth and $\frac{\partial}{\partial n}$ the derivate with respect to the outer normal to the curve. In particular, if u is harmonic, that is $\Delta u = 0$, in a neighborhood of $\overline{\Omega}_m$, then

$$\int_{\partial\Omega_m} \frac{\partial u}{\partial n}\,d\sigma = 0\,.$$

(To see this, just take $v \equiv 1$ and note that harmonic functions are at least twice continously differentiable.) Moreover, if u is a C^2-function in a domain $\Omega \subseteq \mathbf{C}$, then it easily follows from (1) that u is harmonic in Ω if and only

$$\int_{\partial D} \frac{\partial u}{\partial n}\,d\sigma = 0$$

for every small disk D contained in Ω. On the other hand, for harmonic functions u in D, it may be that $\int_\Gamma \frac{\partial u}{\partial n}\,d\sigma \neq 0$ for some closed piecewise smooth curves Γ in D. The following theorem shows that this happens exactly when u does not have a harmonic conjugate.

Theorem 1. *Let Ω be an arbitrary domain in $\mathbf{C}$ and let u be harmonic in Ω. Then a necessary and sufficient condition on u in order that u has a harmonic conjugate in Ω is that for any closed, piecewise smooth curve $\Gamma \subset \Omega$ we have*

$$\int_\Gamma \frac{\partial u}{\partial n}\,d\sigma = 0\,. \tag{2}$$

Proof. As usual, let u_x and u_y denote the partial derivates of u with respect to the variables x and y. Let $t = t(x, y)$ denote the orientated, normalized, tangent vector to the curve Γ at the point (x, y) and $n = n(x, y)$ the normal vector to t, obtained by rotating t by 90 degrees clockwise. Let $g = u_x - iu_y$. Since u is harmonic, g satisfies the Cauchy-Riemann differential equations; hence g is holomorphic in Ω. Let $z = x + iy$ and $dz = dx + i\,dy$. Then

$$\begin{aligned}\int_\Gamma g\,dz &= \int_\Gamma (u_x\,dx + u_y\,dy) + i\int_\Gamma (u_x\,dy - u_y\,dx)\\ &= \int_\Gamma \frac{\partial u}{\partial t} d\sigma + i\int_\Gamma \frac{\partial u}{\partial n}\,d\sigma\,.\end{aligned} \tag{3}$$

Now suppose that u admits a harmonic conjugate v on Ω. Then $f := u + iv$ is holomorphic in Ω and, by using the Cauchy-Riemann differential equations, we see that $g = f'$. Thus g has a primitive and so the integral of g over any closed curve is zero; in particular, by (3), $\int_\Gamma \frac{\partial u}{\partial n}\,d\sigma = 0$.

To prove the converse, fix $z_0 \in \Omega$. For $z \in \Omega$, let Γ_z denote a piecewise smooth curve in Ω joining z_0 with z. Define a function v on Ω by

$$v(z) = \int_{\Gamma_z} \frac{\partial u}{\partial n}\,d\sigma$$

(see also [Ma], vol II, p.146). By our hypothesis, v is well defined. Moreover, $v(z) = \int_{\Gamma_z} (u_x\,dy - u_y\,dx)$. Clearly, v is continuous on Ω. Next we show that the partial derivatives of v exist and that $v_x = -u_y$ and $v_y = u_x$. Hence v is infinitely often differentiable and the pair (u, v) satisfies the Cauchy-Riemann differential equations. Moreover, by Schwarz, it follows that $\Delta v = v_{xx} + v_{yy} = -u_{yx} + u_{xy} \equiv 0$. Hence v is harmonic. Thus v is the harmonic conjugate we were looking for.

Fix $z \in \Omega$ and let h be a nonzero real number, $|h|$ so small that the segment $[z, z+h]$ joining z with $z+h$ is contained in Ω. Look upon z as a point in $\mathbf{R}^2$, say $z = (\xi, \eta)$, and choose the parametrization $\big(x(s), y(s)\big) = (\xi + sh, \eta)$, $(0 \le s \le 1)$, of the segment $[z, z+h]$. Then, as $h \to 0$,

$$\frac{v(z+h) - v(z)}{h} = -\int_0^1 u_y(\xi + sh, \eta)\,ds \to -u_y(\xi, \eta)\,.$$

Thus v_x exists and $v_x = -u_y$. Similarly for v_y. □

For additional reading in the case of a finitely connected planar domain Ω, we refer the reader to a paper of S. Axler [A], where a nice proof of the "Logarithmic Conjugation Theorem" is given: let K_j, $(j = 1, \dots, N)$ be the bounded connected components of $\mathbf{C} \setminus \Omega$ and let u be harmonic in Ω. Then for any choice of points $a_j \in K_j$ there exists real numbers c_j and a function f holomorphic in Ω, such that

$$u(z) = \operatorname{Re} f(z) + \sum_{j=1}^{N} c_j \log|z - a_j| \ (z \in \Omega).$$

For an abstract approach in the context of harmonic forms on Riemannian manifolds, the interested reader could consult [GK].

Finally we note that relation (2) is known in applied mathematics as the *no flux condition* (see e.g. [Ga], p.90), but is often related to the finite connectedness of the domain. We refer the reader to [GR] for typical applications of this condition in the study of conductivity problems or Navier-Stokes equations.

Acknowledgments
I thank Dorin Bucur for several discussions on the theme of this note.

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ХАРМОНИСКИ КОНЈУГИРАНИ ФУНКЦИИ ВО ПРОИИЗВОЛНИ РАМНИНСКИ ОБЛАСТИ

Raymond Mortini

Р е з и м е

Еден од првите резултати во воведниот курс во теоријата на функциите е тврдењето дека испрекината комплексна функција f на рамнинска област Ω има холоморфна примитивна функција акко комплексниот линиски $\int_\Gamma f(z)\,dz = 0$ за секоја затворена крива Γ од областа Ω. Ова е тесно поврзано со прашањето под кои услови хармониска функција u во областа Ω има конјугирано хармониска функција; со други зборови кога постои функција v така што функцијата $u+iv$ е холоморфна во Ω. За да одговориме на ова прашање докажуваме формално сличен критериум: се се заменува f со изводот по нормала $\frac{\partial u}{\partial n}$ од функцијата u по кривата Γ и dz со должината на лакот $d\sigma$.

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