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SOME QUADRATURE FORMULAE FOR LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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Abstract. For the general linear differential equation of the II order with analytical coefficients a(x) and b(x)

$$y'' + a(x)y' + b(x)y = 0 \qquad (1)$$

we can prove the possibility of integration by quadratures in the form of a series of integrals and the general solution is given by (13). So, the Liouville theory for general solution, which is based on the recognition of the fundamental system of particular solutions, can be substituted by quadrature formulae:

$$y = c_1y_1 + c_2y_2 = F(a(x),b(x)) \qquad (2)$$

It means that the recogniton of a fundamental system of particular solutions $\{y_1,y_2\}$ at first is not necessary. We will show that such a fundamental system can be always formed, without trials and guesses. It means that every equation (1) can be solved by quadratures without exclusion and we can reject the traditional claim that in the general case equation (1) cannot be solved by quadratures.

The "solved by quadratures" means not only the finite integrals of coefficients, but also the series of an infinite number of integrals of coefficients.

Idea. Canonical differential equation of the II order

At first we consider a canonical differential equation of the II order

$$y'' + a(x)y = 0 \qquad (3)$$

where a(x) is an analytical function in the neighbourhood of $x_0=0$, which doesn't reduce the generality. It means that a(x), for any $|x| < a$ can be expanded in a power series

$$a(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \qquad (4)$$

where a_i are constants. There is a unique solution, by the Caushy theorem, expanded in the form

$$y = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n + \ldots \qquad (5)$$

where c_i are constants. By differentiating (5) two times and substituting these values with (5) and (4) in (3), we obtain identity

$$2\cdot 1c_2+3\cdot 2c_3x+4\cdot 3c_4x^2+5\cdot 4c_5x^3+\ldots+(n+2)(n+1)c_{n+2}x^n+\ldots+$$
$$+\ldots+(a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots)(c_0+c_1x+c_2x^2+\ldots+c_nx^n+\ldots) \equiv 0$$

Using the method of undetermined coefficients, we obtain

$$\begin{aligned}
c_2 &= \frac{1}{2.1}a_0c_0\\
c_3 &= -\frac{1}{3\cdot 2}(a_0c_1+a_1c_0)\\
c_4 &= -\frac{1}{4\cdot 3}(a_0c_2+a_1c_1+a_2c_0) =\\
&= -\frac{1}{4\cdot 3}[a_0(-\frac{1}{2\cdot 1}a_0c_0)+a_1c_1+a_2c_0]\\
c_5 &= -\frac{1}{5\cdot 4}(a_0c_3+a_1c_2+a_2c_1+a_3c_0) =\\
&= -\frac{1}{5\cdot 4}[a_0(-\frac{1}{3\cdot 2})(a_0c_1+a_1c_0)+a_1(-\frac{1}{2.1}a_0c_0)+a_2c_1+a_3c_0]
\end{aligned} \tag{6}$$

Substituting these values in (5), the solution (5) is expanded in the series with numerical coefficient which depend on known a_k and two arbitrary constants c_0 and c_1. By rearranging the terms, we have

$$\begin{aligned}
y(x) = c_0[1&-\frac{1}{1\cdot 2}a_0x^2-\frac{1}{2\cdot 3}a_1x^3-\frac{1}{3\cdot 4}a_2x^4-\frac{1}{4\cdot 5}a_3x^5-\ldots\\
&+a_0(\frac{a_0}{1\cdot 2\cdot 3\cdot 4}x^4+\frac{a_1}{2\cdot 3\cdot 4\cdot 5}x^5+\frac{a_2}{3\cdot 4\cdot 5\cdot 6}x^6+\ldots)+\\
&+a_1(\frac{a_0}{1\cdot 2\cdot 4\cdot 5}x^5+\frac{a_1}{2\cdot 3\cdot 5\cdot 6}x^6+\ldots)+\\
&+a_2(\frac{a_0}{1\cdot 2\cdot 5\cdot 6}x^6+\frac{a_1}{2\cdot 3\cdot 6\cdot 7}x^7+\ldots)+\ldots]+\\
&+c_1[x-\frac{1}{2\cdot 3}a_0x^3-\frac{1}{3\cdot 4}a_1x^4-\frac{1}{4\cdot 5}a_2x^5-\ldots\\
&+a_0(\frac{a_0}{2\cdot 3\cdot 4\cdot 5}x^5+\frac{a_1}{3\cdot 4\cdot 5\cdot 6}x^6+\ldots)+\\
&+a_1(\frac{a_0}{2\cdot 3\cdot 5\cdot 6}x^6+\frac{a_1}{3\cdot 4\cdot 6\cdot 7}x^7+\ldots)+\ldots]
\end{aligned}$$

and we can see that the series in brackets behind c_0 and c_1 are in fact double integrals of the terms of series (4), i.e.

$$y(x) = c_0[1 - \int_0^x dx\int_0^x a_0 dx - \int_0^x dx\int_0^x a_1 x dx - \int_0^x dx\int_0^x a_2 x^2 dx - \ldots$$
$$+ a_0(\int_0^x dx\int_0^x dx\int_0^x dx\int_0^x a_0 dx + \int_0^x dx\int_0^x dx\int_0^x dx\int_0^x a_1 x dx + \ldots) +$$
$$+ a_1(\int_0^x dx\int_0^x x dx\int_0^x dx\int_0^x a_0 dx + \int_0^x dx\int_0^x x dx\int_0^x dx\int_0^x a_1 x dx + \ldots) + \ldots] +$$
$$+ c_1[x - \int_0^x dx\int_0^x x a_0 dx - \int_0^x dx\int_0^x a_1 x^2 dx - \int_0^x dx\int_0^x a_2 x^3 dx - \ldots$$
$$+ a_0(\int_0^x dx\int_0^x dx\int_0^x dx\int_0^x a_0 x dx + \int_0^x dx\int_0^x dx\int_0^x dx\int_0^x a_1 x^2 dx + \ldots) +$$
$$+ a_1(\int_0^x dx\int_0^x x dx\int_0^x dx\int_0^x a_0 x dx + \int_0^x dx\int_0^x x dx\int_0^x dx\int_0^x a_1 x^2 dx + \ldots) + \ldots]$$

or

$$y(x) = c_0[1 - \int_0^x dx\int_0^x a(x)dx + \int_0^x dx\int_0^x a(x)dx\int_0^x dx\int_0^x a(x)dx -$$
$$- \int_0^x dx\int_0^x a(x)dx\int_0^x dx\int_0^x a(x)dx\int_0^x dx\int_0^x a(x)dx + \ldots] + \tag{7}$$
$$+ c_1[x - \int_0^x dx\int_0^x x a(x)dx + \int_0^x dx\int_0^x a(x)dx\int_0^x dx\int_0^x x a(x)dx -$$
$$- \int_0^x dx\int_0^x a(x)dx\int_0^x dx\int_0^x a(x)dx\int_0^x dx\int_0^x x a(x)dx + \ldots].$$

By induction, we can present the following theorems:

Theorem 1. Canonical differential equation of the II order (3) with analytical coefficient a(x) has one particular solution of the type

$$y_1(x) = 1 - \int_0^x dx\int_0^x a(x)dx + \int_0^x dx\int_0^x a(x)dx\int_0^x dx\int_0^x a(x)dx - \ldots =$$
$$= \sum_{k=0}^{\infty} (-1)^k \underbrace{\int_0^x\int_0^x a(x)dx^2 \ldots \int_0^x\int_0^x a(x)dx^2}_{\text{k-fold double integrals}} \tag{8}$$

Proof. If we differentiate (8) two times, we get

$$y''(x) = -a(x)+a(x)\int_0^x dx\int_0^x a(x)dx - a(x)\int_0^x dx\int_0^x a(x)dx\int_0^x dx\int_0^x a(x)dx + \ldots =$$
$$= -a(x)[1 - \int_0^x dx\int_0^x a(x)dx + \int_0^x dx\int_0^x a(x)dx\int_0^x dx\int_0^x a(x)dx - \ldots] =$$
$$= -a(x)y_1(x),$$

that proves the theorem.

Theorem 2. Canonical differential equation of the II order (3) with analytical coefficient a(x) has the second particular integral of the type

$$y_2(x) = x - \int_0^x dx \int_0^x xa(x)\,dx + \int_0^x dx \int_0^x a(x)\,dx \int_0^x dx \int_0^x xa(x)\,dx - \ldots =$$

$$= x + \sum_{k=1}^{\infty} (-1)^k \underbrace{\int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x a(x)\,dx^2 \ldots \int_0^x\int_0^x xa(x)\,dx^2}_{\text{k-fold double integrals}} \qquad (9)$$

Proof. If we differentiate (9) two times

$$y_2''(x) = -xa(x) + a(x)\int_0^x dx \int_0^x xa(x)\,dx - a(x)\int_0^x dx \int_0^x a(x)\,dx \int_0^x dx \int_0^x xa(x)\,dx + \ldots =$$

$$= -a(x)[x - \int_0^x dx \int_0^x xa(x)\,dx + \int_0^x dx \int_0^x a(x)\,dx \int_0^x dx \int_0^x xa(x)\,dx - \ldots] =$$

$$= -a(x)y_2(x),$$

that proves the theorem.

As for (3), according to the Liouville theorem, the Wronskian determinant is

$$\frac{dW}{dx} = 0, \text{ i.e. } W(y_1,y_2) = \text{const} = c$$

we can determine the constant c using (8) and (9) and the conditions $y_1(0)=1$, $y_1'(0)=0$, $y_2(0)=0$, $y_2'(0)=1$. So

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

and we obtain

Theorem 3. The general solution of equation (3) is

$$y(x) = c_1 \sum_{k=0}^{\infty} (-1)^k \underbrace{\int_0^x\int_0^x a(x)\,dx^2 \ldots \int_0^x\int_0^x a(x)\,dx^2}_{\text{k-double integrals}} + \qquad (10)$$

$$+ c_2[x + \sum_{k=1}^{\infty} (-1)^k \underbrace{\int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x a(x)\,dx^2 \ldots \int_0^x\int_0^x xa(x)\,dx^2}_{\text{k-double integrals}}]$$

So, equation (3) can be always solved by a formula in which the general solution depends only on coefficient a(x) and two arbitrary constants.

Definition. Solution (10) is considered as quadrature in a wider sense, as a series of integrals, because it gives a general solution by means of coefficient a(x) in one single way.

Example. For $a(x)=1$, we obtain $y_1(x)=\cos(x)$, $y_2(x)=\sin(x)$, but for $a(x)=-1$, $y_1(x)=ch(x)$, $y_2(x)=ch(x)$ i.e. we obtain the traditional trigonometry with a constant period and the hyperbolic geometry.

General analytical linear differential equation of the II order

Let us consider the homogeneous differential equation

$$y'' + a(x)y' + b(x)y = 0 \tag{1}$$

where a(x) and b(x) are analytical functions. With substitution

$$y = e^{-\frac{1}{2}\int a(x)\,dx} \cdot z \tag{11}$$

equation (1) can be transformed into a canonical type

$$z'' + (b-\tfrac{1}{4}a^2 - \tfrac{a'}{2})z = 0 \tag{12}$$

On the basis of the previous theorems, we obtain the next

Theorem 4. The general solution of (1) is

$$y(x) = e^{-\frac{1}{2}\int a(x)\,dx}[c_1 \sum_{k=0}^{\infty} (-1)^k \underbrace{\int_0^x \int_0^x (b -\tfrac{1}{4}a^2 -\tfrac{a'}{2})\,dx^2 \ldots \int_0^x \int_0^x (b-\tfrac{1}{4}a^2-\tfrac{a'}{2})\,dx^2}_{\text{k-double integrals}} +$$

$$+ c_2(x+ \sum_{k=0}^{\infty} (-1)^k \underbrace{\int_0^x \int_0^x (b-\tfrac{1}{4}a^2-\tfrac{a'}{2})\,dx^2 \ldots \int_0^x \int_0^x x(b-\tfrac{1}{4}a^2-\tfrac{a'}{2})\,dx^2}_{\text{k-double integrals}})] \tag{13}$$

So, every linear differential equation of the II order can be solved by quadratures in a wider sense.

Note. The effective summation of the series, which is represented by the particular solutions, is not always possible, as we know, because (1) includes a lot of non-elementary and special functions which do not have a finite form. Obviously formula (13) includes solutions of a great number of important equations such as differential equations by Lame', Matie', Hill, elliptical functions and so on.

<u>Application</u>. For the Riccati differential equation

$$y' = a(x)y^2 + b(x)y + c(x) \tag{14}$$

using substitution

$$y = -\frac{1}{a(x)}z$$

we obtain

$$z' = -z^2 + (b+\frac{a'}{a})z - ac,$$

which in turn by substitution

$$z = \frac{u'}{u}$$

can be transformed into

$$u'' - (b+\frac{a'}{a})u' + acu = 0$$

which is of type (1) and it is always possible to find two particular integrals in the type of series of integrals

$$u_1 = e^{1/2\int(b+a'/a)dx} \sum_{k=0}^{\infty} (-1)^k \underbrace{\int_0^x \int_0^x A dx^2 \ldots \int_0^x \int_0^x A dx^2}_{\text{k-double integrals}} \tag{15}$$

and

$$u_2 = e^{1/2\int(b+a'/a)dx} (x + \sum_{k=1}^{\infty} (-1)^k \underbrace{\int_0^x \int_0^x A dx^2 \ldots \int_0^x \int_0^x x A dx^2}_{\text{k-double integrals}}) \tag{16}$$

where

$$A(x) = \frac{1}{2}(b+\frac{a'}{a})' - \frac{1}{4}(b+\frac{a'}{a})^2 + ac . \tag{17}$$

If we come back to the variable $y(x)$, we get the next

<u>Theorem 5</u>. Riccati differential equation (14) has two particular integrals

$$y_1(x) = -\frac{1}{a(x)} \frac{u_1'}{u_1} \tag{18}$$

and

$$y_2(x) = -\frac{1}{a(x)} \frac{u_2'}{u_2} \tag{19}$$

where u_1 and u_2 are given with (15), (16) and (17) and the solution of (14) by the theory is given by

$$\frac{y-y_1}{y-y_2} = ce^{-\int a(x)(y_1-y_2)dx} \tag{20}$$

Conclusion

Theorem 6. Every linear differential equation of the II order can be solved by quadratures in a wider sense.

Theorem 7. Every Riccati equation can be solved by quadratures in a wider sense.

Example. The Riccati equation

$$y' = -y^2 - xy - 2 \tag{21}$$

with substitution

$$y = \frac{u'}{u}$$

is transformed into a linear differential equation of II order

$$u'' + xu' + 2u = 0 \tag{22}$$

and with substitution

$$u = e^{-1/2\int x dx} z = e^{-x^2/4} z$$

is transformed into a canonical type

$$z'' + \left(\frac{3}{2} - \frac{x^2}{4}\right) z = 0$$

with a particular solution

$$z_1 = x - \int_0^x \int_0^x x\left(\frac{3}{2} - \frac{x^2}{4}\right) dx^2 + \int_0^x \int_0^x \left(\frac{3}{2} - \frac{x^2}{4}\right) dx^2 \int_0^x \int_0^x x\left(\frac{3}{2} - \frac{x^2}{4}\right) dx^2 -$$

$$- \int_0^x \int_0^x \left(\frac{3}{2} - \frac{x^2}{4}\right) dx^2 \int_0^x \int_0^x \left(\frac{3}{2} - \frac{x^2}{4}\right) dx^2 \int_0^x \int_0^x x\left(\frac{3}{2} - \frac{x^2}{4}\right) dx^2 + \ldots =$$

$$= x\{1 - \left(\frac{x}{2}\right)^2 + \frac{[(\frac{x}{2})^2]^2}{2!} - \frac{[(\frac{x}{2})^2]^3}{3!} + \ldots\} = xe^{-(x/2)^2} = xe^{-x^2/4}$$

So

$$u = e^{-x^2/4} z_1 = e^{-x^2/4} x e^{-x^2/4} = xe^{-x^2/2}$$

is the particular integral of the linear equation of the II order (22), and

$$y = \frac{u'}{u} = \frac{1-x^2}{x}$$

is the particular integral of Riccati equation (21).

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НЕКОИ КВАДРАТУРНИ ФОРМУЛИ ЗА ЛИНЕАРНИ ДИФЕРЕНЦИЈАЛНИ РАВЕНКИ ОД II РЕД

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Р е з и м е

Ако под квадратура на диференцијална равенка подразбираме алгоритам кој содржи интеграли од коефициентите, во вид на <u>ред од интеграли</u> од коефициентите, тоа

-секоја општа линеарна равенка од II ред може да се реши со квадратури со формулата (13)

-секоја канонична равенка (3) е решлива низ квадратури со формулата (10)

-секоја општа равенка на Riccati (14) е решлива низ квадратури со формулите (15),(16),(17),(18),(19) и (20)

-можна е нова репрезентација на специјалните функции низ интеграли од редови на коефициентите на диференцијалната равенка.

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