Математички Билтен
28 (LIV)
2004 (101–112)
Скопје, Македонија

ISSN 0351–336X

NUMERICAL SOLUTION OF STURM-LIOUVILLE PROBLEMS CONTAININGSPECTRAL PARAMETER IN BOUNDARY OR INTERFACE CONDITIONS

Sonja Gegovska–Zajkova

Abstract

Sturm-Liouville problems containing spectral parameter in the boundary or interface conditions and coefficients which are piecewise functions are considered. Using finite difference method the corresponding discrete spectral problems are obtained. The estimates for the eigenvalues and eigenfunctions which are the solutions of these spectral problems are derived.

1. Statement and Properties of the Problems

We consider a Sturm-Liouville problem:

$$\big(p(x)v'\big)' - q(x)v + \lambda r(x)v = 0, \tag{1.1}$$

where λ is the eigenvalue, $v(x)$ is the eigenfunction, $p(x)$, $q(x)$, $r(x)$ are piecewise continuous functions on $[0,1]$ such that

$$0 < c_1 \le p(x) \le c_2, \quad 0 < c_1 \le r(x) \le c_3, \quad 0 \le q(x) \le c_4, \tag{1.2}$$

$c_1, c_2, c_3, c_4 = \text{const}$.

Let ξ be an interior point in $(0,1)$ where $v(x)$ has to satisfy the condition of conjugation

$$[v]_{x=\xi} = v(x+0) - v(x-0) = 0, \ [pv']_{x=\xi} = -\lambda K v(\xi), \ K = \text{const.} > 0. \tag{1.3}$$

Then $p(x)$ cold have a disconinuity of first order at the point $x = \xi$, $(0 < \xi < 1)$. The boundary conditions could also consist spectral parameter

$$-\alpha_0 v'(0)+\beta_0 v(0)=\lambda\gamma_0 v(0), \quad \alpha_0+\beta_0>0,\ \alpha_0,\beta_0\geq 0,$$
$$\alpha_1 v'(1)+\beta_1 v(1)=\lambda\gamma_1 v(1), \quad \alpha_1+\beta_1>0,\ \alpha_1,\beta_1\geq 0, \tag{1.4}$$
$$\alpha_i,\ \beta_i,\ \gamma_i=\text{const.},\quad i=1,2.$$

Using Dirac distribution, the problem (1.1), along with the conditions (1.3), could be written in the following form:

$$\big(p(x)v'\big)'-q(x)v+\lambda[r(x)+K\delta(x-\xi)]v=0, \tag{1.5}$$

or in the operator form

$$Av=\lambda Bv,$$

letting $H=L_2(0,1)$ and

$$Av=-\big(p(x)v'\big)'+q(x)v, \quad Bv=\big[r(x)+K\delta(x-\xi)\big]v. \tag{1.6}$$

This kind of spectral problems appears, for example, while solving the heat equation with concentrated capacity and combinations of various boundary conditions as a result of using the method of separation of variables [3, 1, 7].

Further, we shall consider the problem (1.1)–(1.4) setting in the boundary conditions (1.4)

$$\alpha_0=\beta_1=\gamma_0=\gamma_1=0, \quad \alpha_1=p(1), \quad \beta_0=1. \tag{1.7}$$

Theorem 1. *The Sturm-Liouville problem* $(1,1)-(1.4)$ *with coefficients given in* (1.7) *has a countable set of eigenvalues* $0<\lambda_1<\lambda_2<\cdots$ *to which correspond the eigenfunctions* $v_1(x), v_2(x), \ldots$. *The eigenfunctions* $\{v_n(x)\}$ *form a complete orthogonal system with respect to a norm* $\|\cdot\|_B$ *which arises from the inner product* $(\cdot,\cdot)_B$, *where*

$$(u,v)_B=\int_0^1 r(x)u\,v\,dx+Ku(\xi)v(\xi). \tag{1.8}$$

Proof. e Let us denote by $\mathcal{K}$ the class of continuous functions $f(x)$ such that $\dfrac{df}{dx}$ is bounded on each of the intervals $(0,\xi)$, $(\xi,1)$. The functions of the class $\mathcal{K}$ satisfy (1.3) and (1.4). We introduce in $\mathcal{K}$ the inner product $(u,v)_B$ as it was done in (1.8). Reinforcing in the norm $\|v\|_B=(v,v)_B^{1/2}$, we obtain a new space $L_{2,B}(0,1)\subset L_2(0,1)$. We introduce new (energy) inner product

$$[u,v]=\int_0^1 (pu'v'+quv)dx \tag{1.9}$$

and define in a usual way the energy space $H_L\subset W_2^1$ with norm $|[v]|=[v,v]^{1/2}$. Thus we obtain the weak formulation of the spectral problem: to find all real eigenvalues λ and eigenfunctions $v(x)$ such that

$$[v, w] = \lambda(v, w)_B, \quad \forall w \in H_L.$$

Using the results given in [5] and Reisz theorem, we can conclude that there exists selfadjoint positive completely continuous (compact) operator $L : L_{2,B}(0,1) \to H_L$ such that $[u, v] = \lambda(v, w)$, $v, w \in H_L$, i.e. $Lv = \lambda v$. Now, the claim of the theorem follows from the theory of linear operators in Hilbert spaces.

If the coefficient $p(x)$, $q(x)$, $r(x)$ are sufficiently smooth, it can be proved that the generalized eigenfunction $v(x)$ is a classical solution of the Sturm-Liouville problem as well.

To prove that a spectral problem $A\,v = \lambda\, B\, v$ can be reduced to the variational form [4]

$$\inf_v \frac{[v, v]}{(v, v)_B} = \lambda_1 \le \lambda_i, \quad i = 2, 3, \ldots, \; v \in H_L,$$

where H_L is energy space equipped with inner product $[u, v]$ and norm $|[u]|^{1/2}$, we set

$$R[v] = \frac{[v, v]}{(v, v)_B} = \frac{\int\limits_0^1 [p(x)v'^2 + q(x)v^2]dx}{\int\limits_0^1 r(x)v^2 dx + Kv^2(\xi)}$$

and $v \in H_L$ is a nonzero element for which $R[v]$ have a minimum value. Then, for $\eta \in H_L$, we have

$$\frac{d}{d\alpha} R[v_1 + \alpha\eta]\Big|_{\alpha=0} = 2\,\frac{[v_1, \eta] - R(v_1, \eta)_B}{(v_1, v_1)_B}.$$

Since η is arbitrary, we can assume that $R[v]$ is the eigenvalue which can be denoted by $\lambda_1 = R[v_1]$, while the corresponding eigenfunction is v_1. From the fact that $R[v]$ reaches minimum value on v_1 in H_L equal to λ_1, it follows that $\lambda_1 < R[v_j] = \lambda_j$ for the eigenvalues and corresponding eigenfunctions such that $j \ne 1$.

The eigenvalues λ_n, $n > 1$ can be founded as a minimum of the functional $R[v]$ in a class of continuous functions $\varphi(x)$ that are defined on $[0, 1]$, they have a piecewise first derivative and satisfy the conditions

$$(\varphi, v_m)_B = \int\limits_0^1 r(x)\varphi v_m\, dx + K\varphi(\xi)v_m(\xi) = 0$$

$$\varphi(0) = 0, \quad p(1)\varphi'(1) = 0, \quad m = 1, 2, \ldots, n-1,$$

where $v_m(x)$ is the $m-$th eigenfunction. This minimum determines the $n-$th eigenvalue

$$\lambda_n = \inf_\varphi R[\varphi] = R[v_n],$$

where v_n is the n–th eigenfunction.

Theorem 2. *The eigenvalues of the problem* (1.1) – (1.4) *with coefficients* (1.7), $\lambda_n \to \infty$, $n \to \infty$, *satisfy the inequalities:*

$$c_5 n^2 \le \lambda_n \le c_6 n^2, \quad c_5 > 0, \tag{1.10}$$

c_5 *and* c_6 *are independent of* c_i, $i = 1, 2, 3, 4$ *and* n.

Proof. By the abstract spectral theory of selfadjoint operators [6], for the operator A it is easy to see that

$$\lambda_1 = \inf_{v \in H_L} \frac{[v,v]}{(v,v)_B} = \inf_{v \in H_L} \frac{\int_0^1 [p(x)v'^2 + q(x)v^2]dx}{\int_0^1 r(x)v^2\,dx + Kv^2(\xi)},$$

$$\lambda_{k+1} = \inf_{\substack{v\,:\,(v,u_i)_B = 0,\\ i = 1,2,\dots,k}} \frac{[v,v]}{(v,v)_B}.$$

Now, we can consider two spectral problems (1.1)–(1.4) with constant coefficient such that $0 < p_1 \le p(x) \le p_2$ and $q_1 \le q(x) \le q_2$, $x \in [0,1]$. Denoting the corresponding eigenfunctions with λ'_k and λ''_k, $k = 1, 2, \dots$, it is easy to see that

$$\lambda'_k \le \lambda_k \le \lambda''_k, \quad k = 1, 2, \dots$$

It is shown in [2] that inequalities (1.10) hold for the eigenvalues λ'_k, λ''_k, $k = 1, 2, \dots$. □

Theorem 3. *The eigenfunctions of the problem* (1.1) – (1.4) *and their derivatives satisfy the inequalities*

$$|v_n(x)| \le c_7, \quad |v'_n(x)| \le c'_8\sqrt{\lambda_n} \le c_8 n, \tag{1.11}$$

where c_7, c_8 *are positive constants independent of* n.

Proof. Further we can assume without loss of generality that functions p, q and r have single discontinuity at the point $x = \xi$, $0 < \xi < 1$ where the conjugation conditions (1.3) hold.

We set in (1.1) $t = \int_0^x r(\zeta)d\zeta$. Then the equation (1.1) with $\lambda = \lambda_n$ takes the form

$$\frac{d}{dt}\left(\overline{p}(t)\frac{d\overline{v}}{dt}\right) - \overline{q}(t)\overline{v} + \lambda[1 + K\delta(x-\xi)]\overline{v} = 0, \quad 0 < t < l, \tag{1.12}$$

$$\overline{v}(0) = 0, \quad \overline{v}'(l) = 0,$$

where $\overline{p}(t) = p(x)r(x)$, $\overline{q}(t) = \dfrac{q(x)}{r(x)}$, $\overline{v}(t) = v(x)$, $l = \int_0^1 r(x)dx$.

For $0 < x < \xi$ we have

$$-(\overline{p}(t)\overline{v}')' + \overline{q}(t)\overline{v} = \lambda\overline{v}, \quad \overline{v}(0) = 0.$$

We multiply this equation with $\overline{v}'$ and integrate from 0 to t:

$$-\int_0^t (\overline{p}(t_1)\overline{v}')'\,\overline{v}'\,dt_1 + \int_0^t \overline{q}(t_1)\overline{v}\,\overline{v}'\,dt_1 = \lambda\int_0^t \overline{v}\,\overline{v}'\,dt_1 .$$

Taking into account that

$$(\overline{p}\,\overline{v}')'\overline{v}' = \frac{1}{2\overline{p}}[(\overline{p}\,\overline{v}')^2]', \quad \overline{v}\,\overline{v}' = \frac{1}{2}(\overline{v}^2)',$$

after integration by parts we obtain

$$\begin{aligned}\overline{p}(0)\,[\overline{v}'(0)]^2 + \lambda\,\overline{v}^2(0) =&\overline{p}(t)\,[\overline{v}'(t)]^2 + \lambda\,\overline{v}^2(t) \\ &- \int_0^t \overline{p}'(t_1)\,[\overline{v}'(t_1)]^2\,dt_1 + 2\int_0^t \overline{q}(t_1)\,\overline{v}(t_1)\,\overline{v}'(t_1)\,dt_1 .\end{aligned} \tag{1.13}$$

No, we are going to estimate integrals on the right hand side of (1.13). First we multiply (1.12) with $\overline{v}(t)$ and integrate from 0 to l:

$$-\int_0^l \left[\frac{d}{dt}\left(\overline{p}(t)\frac{d\overline{v}}{dt}\right)\overline{v} + \overline{q}(t)\overline{v}^2\right]dt = \lambda_n\left(\int_0^l \overline{v}^2 dt + K\overline{v}^2(\xi)\right).$$

Integrating by parts, taking into account that $\int_0^l \overline{v}^2(t)dt + K\overline{v}^2(\xi) = 1$ and the boundary conditions, we obtain

$$\int_0^l \overline{p}(t)\left(\frac{d\overline{v}}{dt}\right)^2 dt + \int_0^l \overline{q}(t)\overline{v}^2 dt = \lambda_n\left(\int_0^l \overline{v}^2 dt + K\overline{v}^2(\xi)\right),$$

which implies the inequality

$$\int_0^l \overline{p}(t)\left(\frac{d\overline{v}}{dt}\right)^2 dt \le \lambda_n .$$

Using this inequality and integrating once again from 0 to l we found that

$$\begin{aligned}2\left|\int_0^l dt\int_0^t \overline{q}(t_1)\overline{v}\,\overline{v}'\,dt_1\right| &\le 2l\|\overline{q}\|_C\left[\int_0^l \overline{v}^2(t)dt\right]^{1/2}\left[\int_0^l (\overline{v}'(t)^2 dt\right]^{1/2} \\ &\le 2l\|\overline{q}\|_C\sqrt{\frac{\lambda_n}{c_1}} = c_9\sqrt{\lambda_n},\end{aligned}$$

since

$$\int_0^l (\overline{v}')^2 dt \le \frac{1}{c_1}\int_0^l \overline{p}(t)\,\overline{v}'^2 dt \le \frac{\lambda_n}{c_1}.$$

On the other hand

$$\left|\int_0^l dt\int_0^t \overline{p}'(t_1)(\overline{v}')^2\,\frac{\overline{p}(t_1)}{\overline{p}(t_1)}dt_1\right| \le l\left\|\frac{\overline{p}'}{\overline{p}}\right\|_{C}\int_0^l \overline{p}(t)(v')^2 dt \le c_{10}\lambda_n .$$

Thus, the following estimate holds

$$\overline{p}(0)[\overline{v}'(0)]^2+\lambda_n\overline{v}^2(0)\le \overline{p}(t)[\overline{v}'(t)]^2+\lambda_n\overline{v}^2(t)+c_9\sqrt{\lambda_n}+c_{10}\lambda_n .$$

Integrating the last inequality from 0 to l we obtain

$$l[\overline{p}(0)[\overline{v}'(0)]^2+\lambda_n\overline{v}^2(0)]\le \int_0^l \overline{p}(t)[\overline{v}'(t)]^2dt+\lambda_n\Big(\int_0^l \overline{v}^2(t)dt+ K\overline{v}^2(\xi)\Big)+l(c_9\sqrt{\lambda_n}+c_{10}\lambda_n),$$

or

$$\overline{p}(0)[\overline{v}'(0)]^2+\lambda\overline{v}^2(0)\le \overline{c}_{10}\lambda_n+c_9\sqrt{\lambda_n},\quad \overline{c}_{10}=c_{10}+\frac{2}{l}.$$

Then

$$\begin{aligned}\overline{p}(\xi)[\overline{v}'(\xi)]^2+\lambda_n\overline{v}^2(\xi) \le&\overline{p}(0)[\overline{v}'(0)]^2+\lambda_n\overline{v}_n(0)+c_9\sqrt{\lambda_n}\\ &+\overline{c}_{10}\lambda_n\le \tilde{c}_{10}\lambda_n+\tilde{c}_9\sqrt{\lambda_n}.\end{aligned}$$

Similar estimate holds when $\xi<x<l$. In such a way, we obtain that

$$\overline{v}_n^2(t)\le \frac{\tilde{c}_9}{\sqrt{\lambda_n}}+\tilde{c}_{10}\le c_7^2 \;\Rightarrow\; |\overline{v}_n(t)|=|v_n(x)|\le c_7,$$
$$|\overline{v}_n'(t)|=\frac{1}{r(x)}|v_n'(x)|\le c_{11}\lambda_n \;\Rightarrow\; |v_n'(x)|\le c_8'\sqrt{\lambda_n},$$

which completes the proof of the theorem. □

2. Formulation and Properties of the Difference Problems

Let $\overline{\omega}_h=\{x_i=ih,\, i=0,1,\dots,N,\, hN=1\}$ is a uniform mesh in $[0,1]$ chosen so that $\xi=x_m$ is a node. We approximate the problem (1.1)–(1.4) with coefficients (1.7) on the mesh $\overline{\omega}_h$ by the difference scheme

$$
\begin{aligned}
&-(ay_{\overline{x}})_x + dy = \lambda^h \left(\rho(x) + K\delta_h(x-\xi)\right) y, \quad x \in \omega_h, \\
&y_0 = 0, \qquad ay_{\overline{x}} + \frac{h}{2} dy = \frac{h}{2}\lambda^h \rho y, \quad x = x_N,
\end{aligned}
\tag{2.1}
$$

where

$$
\delta_h(x-\xi) = \begin{cases} 0, & x \in \omega_h \backslash \{\xi\} \\ \dfrac{1}{h}, & x = \xi \end{cases},
$$

or

$$
\begin{aligned}
&-(ay_{\overline{x}})_x + dy = \lambda^h \rho\, y, && 0 < x = x_i = ih < 1,\ i \neq m \\
&-\frac{h}{\overline{K}}(ay_{\overline{x}})_x + \frac{h}{\overline{K}}\overline{d}\, y = \lambda^h y, && x = x_m,\ \overline{K} = K + h\rho(\xi),
\end{aligned}
$$

$$
\overline{d} = \begin{cases} \dfrac{d(x_m - 0) + d(x_m + 0)}{2}, & x = x_m = \xi \\ d, & x \in \omega_h \setminus \{\xi\} \end{cases}
$$

$$
y_0 = 0, \qquad ay_{\overline{x}} + \frac{h}{2} dy = \frac{h}{2}\lambda^h \rho y, \qquad x = x_N.
$$

We shall denote by $\Lambda\, y$ difference operator

$$
\Lambda\, y = \begin{cases} -(ay_{\overline{x}})_x + dy, & x = x_i,\ i = 1, 2, \dots, N-1 \\ \dfrac{2}{h} ay_{\overline{x}} + dy, & i = N \\ 0, & i = 0 \end{cases}
$$

while

$$
\begin{aligned}
a(x) &= \frac{p(x) + p(x-h)}{2}, \qquad \text{for } x \neq \xi, \xi + h, \\
a(\xi) &= \frac{p(\xi - 0) + p(\xi - h)}{2}, \\
a(\xi + h) &= \frac{p(\xi + h) + p(\xi + 0)}{2} \qquad \text{and} \\
d(x_i) &= q(x_i), \quad \rho(x_i) = r(x_i).
\end{aligned}
$$

Thus the following inequalities hold:

$$
0 < c_1 \le a \le c_2, \quad 0 < c_1 \le \rho(x) \le c_3, \quad 0 \le d(x) \le c_4. \tag{2.2}
$$

Now, the difference Sturm-Liouville problem reads as follows: to find nontrivial solutions of the problem (2.2) (the eigenfunctions) which correspond to the values of the parameter λ^h (the eigenvalues).

Theorem 4. *There exist $N-1$ eigenvalues of the problem* (2.1), $0 < \lambda_1^h < \cdots < \lambda_{N-1}^h$ *to which correspond the eigenfunctions*

$y_1(x), \ldots, y_{N-1}(x)$. The eigenfunctions $\{y_n(x)\}$ form an orthogonal system in the l^2_{N-1} space with scalar product

$$(y,v]_{B_h} = \sum_{\substack{i=1\\ i\neq m}}^{N-1} \rho_i y_i v_i h + \overline{K} y_m v_m + \frac{h}{2}\rho_N y_N v_N, \quad \overline{K} = K + h\rho(\xi). \quad (2.3)$$

Proof. Operator Λ is selfadjoint and positive definite in the scalar product (2.3). We will show that

$$(\Lambda y, v]_* = (a y_{\overline{x}}, v_{\overline{x}}] + (\overline{d} y, v]_*,$$

where y and v are discrete functions that satisfy boundary conditions. Here

$$(y,v] = \sum_{i=1}^{N} y_i\, v_i\, h, \quad (y,v]_* = \sum_{i=1}^{N-1} y_i\, v_i\, h + \frac{h}{2} y_N\, v_N.$$

Using Green's formula, we obtain

$$\begin{aligned}(\Lambda\, y, v]_* &= -\sum_{i=1}^{N-1}((a y_{\overline{x}})_{x,i} - \overline{d}_i\, y_i)\, v_i\, h + \frac{h}{2}\left(\frac{2}{h} a y_{\overline{x},N} + \overline{d}_N\, y_N\right) v_N \\ &= -\left((a y_{\overline{x}})_x, v\right) + (\overline{d} y, v) + a_N\, y_{\overline{x},N}\, v_N + \frac{h}{2}\overline{d}_N\, y_N\, v_N \\ &= (a y_{\overline{x}}, v_{\overline{x}}] + (\overline{d} y, v) + \frac{h}{2}\overline{d}_N\, \rho_N\, y_N\, v_N \\ &= (a y_{\overline{x}}, v_{\overline{x}}] + (\overline{d} y, v]_*.\end{aligned}$$

The assertion of the theorem follows from selfadjointness and positive definiteness of the operator Λ. □

The difference problem (2.1) is equivalent to a variational problem: to find a minimum of the functional $D_N[\varphi]$ in a class of mesh functions satisfying the boundary conditions and $H_N[\varphi] = 1$,

$$\lambda^h = \frac{D_N[y]}{H_N[y]},$$

where

$$D_N[y] = (a\,, y^2_{\overline{x}}] + (\overline{d}\, y, y]_*, \quad H_N[y] = (y,y]_{B_h}.$$

Let us denote the smallest eigenvalue $\lambda_1^h = D_N[y_1]$, where y_1 is a corresponding eigenfunction of the problem (2.1). We can find the n–th eigenvalue, $n > 1$, as a minimum of the functional $D_N[\varphi]$ in a class of functions satisfying the boundary conditions and condition of orthogonality

$$H_N[\varphi, y^{(l)}] = (\varphi, y^{(l)}]_{B_h} = 0, \quad l = 1, 2, \ldots, n-1.$$

Here $y^{(l)}$ is the l–th eigenfunction.

The differential Sturm-Liouville problem is algebraic one. Further, we will prove the following assertions.

Theorem 5. *The eigenvalues of the problem* (2.1) *satisfy inequalities*

$$M_1' n^2 \le \lambda_n^h \le M_2' n^2, \quad n = 1, 2, \ldots, N-1, \tag{2.5}$$

where M_1' and M_2' are positive constants independent of h and n.

Proof. First we shall consider special case setting $a \equiv 1$, $d \equiv 0$, $\rho \equiv 1$, and ξ is rational. Thus we obtain the following discrete problem:

$$-y_{\bar{x}x} = \lambda^h y, \quad x \ne x_m,$$

$$-y_{\bar{x}x,m} = \lambda^h \left(1 + \frac{K}{h}\right) y_m, \quad x = x_m,$$

$$y_0 = 0, \quad y_{\bar{x},N} = \frac{h}{2}\lambda^h y_N.$$

The eigenvalues of the discrete problems are

$$0 < \lambda_1^h = \frac{4}{h^2}\sin^2\frac{\alpha_1 h}{2} < \lambda_2^h < \cdots < \lambda_{N-1}^h = \frac{4}{h^2}\sin^2\frac{\alpha_{N-1} h}{2},$$

while corresponding eigenfunctions could bi written in the explicit form

$$y_i(x) = \begin{cases} \cos\alpha_i^h(1-\xi)\sin\alpha_i^h x, & 0 < x < \xi \\ \sin\alpha_i^h \xi \cos\alpha_i^h(1-x), & \xi < x < 1 \end{cases},$$

where α_i^h, $i = 1, 2, \ldots, N-1$ are the first positive roots of the equation (see fig.1 and fig 2)

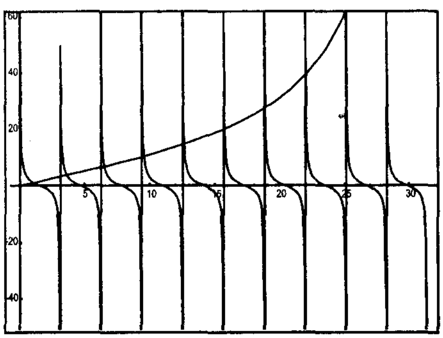

Fig. 1: $h=10^{-1}$, $K=1$, $\xi=0.5$

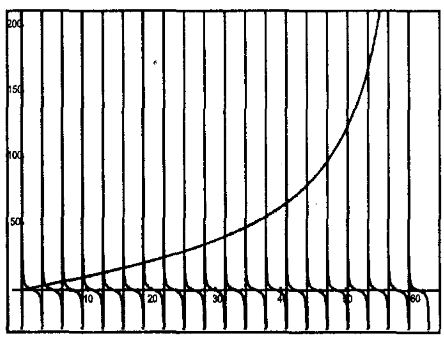

Fig. 2: $h=5*10^{-2}$, $K=1$, $\xi=0.5$

$$\frac{2K}{h}\tan\frac{\alpha^h h}{2} = \cot\alpha^h\xi - \tan\alpha^h(1-\xi).$$

It is easy to see that $\alpha^h \to \alpha$ when $h \to 0$. On the other hand we can set

$$\overset{\circ}{\lambda}{}^h_n = \frac{4}{h^2}\sin^2\frac{\alpha^h_n h}{2} = (\alpha^h_n)^2 f(\zeta_n), \quad f(\zeta_n) = \frac{\sin^2\zeta_n}{\zeta_n^2}, \quad \zeta_n = \frac{\alpha^h_n h}{2}.$$

Since $f(\zeta_n)$ is monotonic decreasing function in $\left[0, \frac{\pi}{2}\right]$, the following estimate holds:

$$\frac{4}{\pi^2} < f(\zeta_n) < 1, \quad 0 < \zeta_n < 1,$$

which implies

$$\frac{4(\alpha^h_n)^2}{\pi^2} < \overset{\circ}{\lambda}{}^h_n < (\alpha^h_n)^2. \tag{2.6}$$

Thus, the inequalities (1.10) hold for $\overset{\circ}{\lambda}{}^h_n$ since $\alpha^h_n \to \alpha$ when $h \to 0$.

The following inequalities are obvious:

$$\frac{D_N[\varphi]}{H_N[\varphi]} = \frac{(a,(\varphi_{\bar{x}})^2] + (\bar{d},\varphi^2]_*}{(1,\varphi^2]_{B_h}} \le c_2\,\overset{\circ}{\lambda}{}^h_n + \frac{c_4}{c_1};$$

$$\frac{D_N[\varphi]}{H_N[\varphi]} \ge c_1\frac{(1,(\varphi_{\bar{x}})^2]}{(1,\varphi^2]_{B_h}} + \frac{(\bar{d},\varphi^2]_*}{(1,\varphi^2]_{B_h}} \ge c_1\,\overset{\circ}{\lambda}{}^h_n$$

i.e.

$$c_1\,\overset{\circ}{\lambda}{}^h_n \le \lambda^h_n \le c_2\,\overset{\circ}{\lambda}{}^h_n + \frac{c_4}{c_1}.$$

Combining this inequalities with (2.6) we obtain the estimate (2.5). □

Theorem 6. *For the eigenfunctions of the problem* (2.1) *the following estimates hold*

$$\|y_n\|_C \le M_1\sqrt{n}, \quad \|(y_n)_{\bar{x}}\|_C \le M_2 n^{3/2}, \tag{2.7}$$

where $\|y\|_C = \max_{x\in\omega_h}|y(x)|$, $\|y_{\bar{x}}\|_C = \max_{1\le i\le N}|y_{\bar{x},i}|$, M_1, M_2 *are constants independent of* h *and* n.

Proof. Let $y = y_n$ be the eigenfunction that corresponds to eigenvalue $\lambda^h = \lambda_n^h$ of the problem (2.1) and x, x' are arbitrary mesh points. First we shall assume the obvious identities

$$y^2(x) - y^2(x') = \sum_{s=x'+h}^{x} [y(s) - y(s-h)] y_{\bar{s}} h, \tag{2.8}$$

$$\begin{aligned}&\big(a(x) y_{\bar{x}}(x)\big)^2 - (a(x') y_{\bar{x}}(x'))^2 = \\ &-\sum_{s=x'}^{x-h} [d(s) - \lambda^h \rho(s)][a(s) y_{\bar{s}}(s) + a(s+h) y_s(s) y(s)]\, h.\end{aligned} \tag{2.9}$$

The normality condition $(y, y]_{B_h} = 1$ implies that there exist at least one point x' such that $y^2(x') \leq 1$. Applying the Cauchy-Schwartz's inequality on the right hand side of (1.10), according to (2.4) and (1.10) we establish that

$$y^2(x) \leq 1 + \frac{2}{\sqrt{c_1}} \sqrt{\lambda}^h \leq M_1^2 n,$$

i.e.

$$\|y_n\|_C \leq M_1 \sqrt{n}.$$

It follows from the condition $(a, (y_{\bar{x}})^2] \leq \lambda^h$ that there exists point x' such that $a(x') y_{\bar{x}}^2(x') \leq \lambda^h$ and $[a(x') y_{\bar{x}}(x')]^2 \leq c_2 \lambda^h$. We apply once again Cauchy-Schwartz's inequality on the right hand side of (2.9) and combine the obtained term with (1.2), (2.4) and (1.10). Thus

$$y_{\bar{x}}^2(x) \leq \frac{c_2}{c_1^2} \lambda^h + \frac{2c_4}{c_1^2} \sqrt{c_2} \sqrt{\lambda^h} + \frac{2}{c_1^2} \sqrt{c_2 c_3} (\lambda^h)^{3/2} \leq M_2^2 n^3.$$

Since x is arbitrary, we complete the proof of the theorem. □

References

[1] Belinskiy, B. P., Dauer, J. F.: *Eigenoscillations of mechanical systems with boundary conditions containing the frequency,* Q.uarterly of Appl.Math. 16 (1999), 521–541.

[2] С. Геговска–Зајкова: *Спектрални проблеми кои содржат сингуларни дистрибуции и нивна примена,* Докторска дисертација, Скопје 2004.

[3] B. S. Jovanović, L. G. Vulkov: *On the convergence of finite difference schemes for the heat equation with concentrated capacity,* Numer. Math. 89, No 4 (2001), 715–734.

[4] Марчук, Г. И., Агошков, В. И.: *Введение в проекционно сеточные методы,* Наука, Москва (1981).

[5] Б.П. Михайлов. *Дифференциальные уравнения в частных производных,* Наука, Москва (1976).

[6] A. A. Samarskii: *Theory of difference schemes.* Nauka, Moscow 1989 (in Russian).

[7] A. N. Tihonov, A.A. Samarskii: *Equations of Mathematical Physics,* GITTL, Moscow 1953 (in Russian).

НУМЕРИЧКО РЕШЕНИЕ НА ШТУРМ-ЛИУВИЛОВИ ПРОБЛЕМИ КОИ СОДРЖАТ СПЕКТРАЛЕН ПАРАМЕТАР ВОГРАНИЧНИТЕ ИЛИ УСЛОВИТЕ ЗА КОНЈУГАЦИЈА

Соња Геговска–Зајкова

Резиме

Во трудот се разгледувани проблеми на Штурм-Лиувил кои содржат спектрален параметар на границите или во условите за конјугација и коефициенти кои припаѓаат на класата по делови глатки функции. Дадени се оцени за сопствените вредности и сопствените функции кои се решенија на овие проблеми. Конструирани се диференцни шеми за нумеричко решавање на разгледуваните проблеми и се изведени соодветни оцени за дискретните сопствени вредности и сопствени функции.

University "St. Kiril and Metodij"
Faculty of Electrical Engineering
P.O.Box 574
1000 Skopje
Republic of Macedonia

e-mail: szajkova@etf.ukim.edu.mk