Математички Билтен
21 (XXI)
1997 (103–110)
Скопје, Македонија

ISSN 0351–336X

l^∞ AS n–NORMED SPACE

Aleksa Malčeski

Abstract

The concept of a n–norm on the vector space with dimension greater than n, $n > 1$, was introduced by A. Misiak ([4]). It is multidimensionl analogy of the concept of a norm. In [1], [2], [3] and [4] was proved several properties of the n–normed spaces. In this work we will prove that the space of the bounded real sequences with usual operations adding and product with scalar is a real n–normed space.

Let L be a real vector space with dimension greater than n, $n > 1$ and $\|\bullet\ \ldots, \bullet\|$ is a real functionon on L^n which satisfy the following conditions:

i) $\|x_1, \ldots, x_n\| \geq 0$, for every $x_1, \ldots, x_n \in L$ and $\|x_1, \ldots, x_n\| = 0$ if and only if the set $\{x_1, \ldots, x_n\}$ is lineary dependent.

ii) $\|x_1, \ldots, x_n\| = \|\pi(x_1), \ldots, \pi(x_n)\|$, for every $x_1, \ldots, x_n \in L$ and for each bejection $\pi\{x_1, \ldots, x_n\} \to \{x_1, \ldots, x_n\}$

iii) $\|\alpha x_1, \ldots, x_n\| = |\alpha| \cdot \|x_1, \ldots, x_n\|$, for every $x_1, \ldots, x_n \in L$ and every $\alpha \in R$

iv) $\|x_1 + x_1', \ldots, x_n\| \leq \|x_1, \ldots, x_n\| + \|x_1', \ldots, x_n\|$, for every $x_1, \ldots, x_n, x_1' \in L$.

We call the function $\|\bullet, \ldots, \bullet\|$ a n–norm on L_1 and we call $(L, \|\bullet, \ldots, \bullet\|)$ n–normed space.

We denote with l^∞ the set of all bounded sequences of real numbers. These set with usuall operations adding sequences and product with real number is a real vector space, ([15]). We will prove the following lemma.

Lemma 1. The vectors $x_i = (x_{ij})_{j=1}^{\infty} \in l^\infty$, $i = 1, 2, \ldots, k$; $k \in N$, are lineary dependent if and only if:

$$\begin{vmatrix} x_{1j_1} & x_{1j_2} & \cdots & x_{1j_{k-1}} & x_{1j_k} \\ x_{2j_1} & x_{2j_2} & \cdots & x_{2j_{k-1}} & x_{2j_k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{kj_1} & x_{kj_2} & \cdots & x_{kj_{k-1}} & x_{kj_k} \end{vmatrix} = 0, \tag{1}$$

for every natural numbers $j_1 < j_2 < \cdots < j_k$.

Proof. If the vectors $x_i = (x_{ij})_{j=1}^{\infty} \in l^\infty$, $i = 1, 2, \ldots, k$ are lineary dependent, then (1) is obviously.

The converse statement will be prove by induction in k.

Let $k = 2$ and let the vectors $x_i = (x_{ij})_{j=1}^{\infty} \in l^\infty$, $i = 1, 2$ satisfy:

$\begin{vmatrix} x_{1j_1} & x_{1j_2} \\ x_{2j_1} & x_{2j_2} \end{vmatrix} = 0$, for every natural numbers $j_1 < j_2$, which means

$$x_{1j_1}x_{2j_2} - x_{1j_2}x_{2j_1} = 0, \text{ for every natural numbers } j_1 < j_2, \tag{2}$$

If $x_{1m} = 0$, for every $m \in N$, then $x_1 = 0$, and so $x_1 = 0 \cdot x_2$, which means that the vectors $x_i = (x_{ij})_{j=1}^{\infty} \in l^\infty$, $i = 1, 2$ are lineary dependent. If there exist $m \in N$ such that $x_{1m} \neq 0$, then from (2) follows $x_{2p} = \dfrac{x_{2m}}{x_{1m}} x_{1p}$, for every natural number p. These means that the vectors $x_i = (x_{ij})_{j=1}^{\infty} \in l^\infty$, $i = 1, 2$, are lineary dependent.

Suppose thet the statement is true for some $k \geq 2$ and that the vectors $x_i = (x_{ij})_{j=1}^{\infty} \in l^\infty$, $i = 1, 2, \ldots, k+1$ satisfies:

$$\begin{vmatrix} x_{1j_1} & x_{1j_2} & \cdots & x_{1j_k} & x_{1j_{k+1}} \\ x_{2j_1} & x_{2j_2} & \cdots & x_{2j_k} & x_{2j_{k+1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{k+1j_1} & x_{k+1j_2} & \cdots & x_{k+1j_k} & x_{k+1j_{k+1}} \end{vmatrix} = 0,$$

for every natural numbers $j_1 < j_2 < \cdots < j_k < j_{k+1}$, which means that for every natural numbers $j_1 < j_2 < \cdots < j_k < j_{k+1}$ it is true:

$$\sum_{i=1}^{k+1} (-1)^{k+1+i} x_{ij_{k+1}} \begin{vmatrix} x_{1j_1} & x_{1j_2} & \cdots & x_{1j_k} \\ \cdots & \cdots & \cdots & \cdots \\ x_{i-1j_1} & x_{i-1j_2} & \cdots & x_{i-1j_k} \\ x_{i+1j_1} & x_{i+1j_2} & \cdots & x_{i+1j_k} \\ \cdots & \cdots & \cdots & \cdots \\ x_{k+1j_1} & x_{k+1j_1} & \cdots & x_{k+1j_1} \end{vmatrix} = 0 \qquad (3)$$

We will consider two cases:

– For every natural numbers $j_1 < j_2 < \cdots < j_k < j_{k+1}$ all determinants in (3) are equal zero. Then, for every k vectors of the set of vectors

$$x_i = (x_{ij})_{j=1}^{\infty} \in l^{\infty}, \quad i = 1, \ldots, k+1$$

the conditions (11) is true.

By the inductive hipotheses these vectors are lineary dependent. These means that the vectors

$$x_i = (x_{ij})_{j=1}^{\infty} \in l^{\infty}, \quad 1, \ldots, k+1$$

are lineary dependent.

– For every natural numbers $j_1 < j_2 < \cdots < j_k < j_{k+1}$ some of the determinants in (3) is different of zero. Withous loss of generality we may assume that

$$\begin{vmatrix} x_{1j_1} & x_{1j_2} & \cdots & x_{1j_{k-1}} & x_{1j_k} \\ x_{2j_1} & x_{2j_2} & \cdots & x_{2j_{k-1}} & x_{2j_k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{kj_1} & x_{kj_2} & \cdots & x_{kj_{k-1}} & x_{kj_k} \end{vmatrix} \neq 0 .$$

If for $i = 1, 2, \ldots, k$ we put

$$\alpha_i = (-1)^{k+i} \frac{\begin{vmatrix} x_{1j_1} & x_{1j_2} & \cdots & x_{1j_{k-1}} & x_{1j_k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{i-1j_1} & x_{i-1j_2} & \cdots & x_{i-1j_{k-1}} & x_{i-1j_k} \\ x_{i+1j_1} & x_{i+1j_2} & \cdots & x_{i+1j_{k-1}} & x_{i+1j_k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{k+1j_1} & x_{k+1j_2} & \cdots & x_{k+1j_{k-1}} & x_{k+1j_k} \end{vmatrix}}{\begin{vmatrix} x_{1j_1} & x_{1j_2} & \cdots & x_{1j_{k-1}} & x_{1j_k} \\ x_{2j_1} & x_{2j_2} & \cdots & x_{2j_{k-1}} & x_{2j_k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{kj_1} & x_{kj_2} & \cdots & x_{kj_{k-1}} & x_{kj_k} \end{vmatrix}}$$

then we may write the equality (3) in the following form:

$$x_{k+1j_{k+1}} = \sum_{i=1}^{k} \alpha_1 x_{ij_{k+1}} .$$

From the condition of the Lemma 1 and the properties of the determinants it follows that for the natural numbers $j_1 < j_2 < \cdots < j_k$ and every naturall m it is true:

$$\begin{vmatrix} x_{1j_1} & x_{1j_2} & \cdots & x_{1j_k} & x_{1m} \\ x_{2j_1} & x_{2j_2} & \cdots & x_{2j_k} & x_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{k+1j_1} & x_{k+1j_2} & \cdots & x_{k+1j_k} & x_{k+1m} \end{vmatrix} = 0 .$$

Now, as in the previous considerations can be prove that for every natural number m:

$$x_{k+1m} = \sum_{i=1}^{k} \alpha_i x_{im} ,$$

which means that the vectors $x_i = (x_{ij})_{i=1}^{\infty} \in l^{\infty}$, $i = 1, 2, \ldots, k+1$ are lineary dependent.

Note. In the same way can be prove that the condition of the Lemma 1 is necessary and sufficient for lineary dependence in:

– the space $l_2 = \left\{ x \mid x = (x_i)_{i=1}^{\infty} ,\ x_i \in R,\ \sum_{i=1}^{\infty} x_i^2 < \infty \right\}$, with usual operations;

- the space c of all convergent sequences $x = (x_i)_{i=1}^{\infty}$, with usual operations;
- the space c_0 of all sequences $(x_i)_{i=1}^{\infty}$ whcich converg to zero with usual operations; and
- the space R^{∞} of all real sequences, with usual operations.

Let $x_i = (x_{ij})_{j=1}^{\infty} \in l^{\infty}$, $i = 1, 2, \ldots, n$. With

$$\|x_1, \ldots, x_n\| = \sup_{\substack{j_1, \ldots, j_n \in N \\ f_1 < \cdots < f_n}} \left\| \begin{matrix} x_{1j_1} & x_{1j_2} & \cdots & x_{1j_{n-1}} & x_{1j_n} \\ x_{2j_1} & x_{2j_2} & \cdots & x_{2j_{n-1}} & x_{2j_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{nj_1} & x_{nj_2} & \cdots & x_{nj_{n-1}} & x_{nj_n} \end{matrix} \right\| \tag{4}$$

we define a function $\|\bullet, \ldots, \bullet\|: l^{\infty} \times \ldots \times l^{\infty} \to R$. Since $x_i = (x_{ij})_{j=1}^{\infty} \in l^{\infty}$, $i = 1, 2, \ldots, n$ there exist constants M_i, $i = 1, 2, \ldots, n$ such that $|x_{ij}| \leq M_i$, for every $j \in N$, which implies

$$\|x_1, \ldots, x_n\| \leq n! \prod_{i=1}^{n} M_i \,,$$

These means that function $\|\bullet, \ldots, \bullet\|$ is good defined.

Lemma 2. $(l^{\infty}, \|\bullet, \ldots, \bullet\|)$ is a real n–normed space.

Proof. Since the function $\|\bullet, \ldots, \bullet\|$ is good defined, we have prove that it satisfies the axioms of the n–norm.

From the definition of $\|\bullet, \ldots, \bullet\|: l^{\infty} \times \ldots \times l^{\infty} \to R$ it follows that $\|x_1, \ldots, x_n\| \geq 0$. It is clear that $\|x_1, \ldots, x_n\| = 0$ if and only if

$$\left| \begin{matrix} x_{1j_1} & x_{1j_2} & \cdots & x_{1j_{n-1}} & x_{1j_n} \\ x_{2j_1} & x_{2j_2} & \cdots & x_{2j_{n-1}} & x_{2j_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{nj_1} & x_{nj_2} & \cdots & x_{nj_{n-1}} & x_{nj_n} \end{matrix} \right\| = 0 \,,$$

for every natural numbers $j_1, \ldots, j_n$ such that $j_1 < \cdots < j_n$. From Lemma 1 we have that $\|x_1, \ldots, x_n\| = 0$ if and only if the vectors $x_1, \ldots, x_n$ are lineary dependent.

The properties ii), iii) and iv) of the definition of the n–norm follows from the properties of the determinants and the properties of supremum.

In the end of these work we give some notes on the space l^{∞}.

1. In [1] was proved that every n–normed space can be introduced topology τ which make the space into a local convex space. In these topology the n–norm is continuous in each variable. We have a question: if in l^∞ we introduced a topology τ in the descripted way, what properties has the topological space (l^∞, τ).

2. In [2] was given the following definition for a strong convex n–normed space: the n–normed vector space $(L, \|\bullet, \ldots, \bullet\|)$ is called strong convex if from

$$\|a+b, x_1, x_2, \ldots, x_{n-1}\| = \|a, x_1, x_2, \ldots, x_{n-1}\| + \|b, x_1, x_2, \ldots, x_{n-1}\|;$$

$$\|a, x_1, x_2, \ldots, x_{n-1}\| = \|b, x_1, x_2, \ldots, x_{n-1}\| = 1$$

and

$$P(a, b) \cap P(x_1, x_2, \ldots, x_{n-1}) = \{0\}$$

follows that $a = b$.

If we use

$$a = \left(1 - \frac{1}{2}, 1 - \frac{1}{2^2}, 1 - \frac{1}{2^3}, 1 - \frac{1}{2^4}, \ldots, 1 - \frac{1}{2^n}, \ldots\right);$$

$$b = \left(0, 1 - \frac{1}{2}, 1 - \frac{1}{2^2}, 1 - \frac{1}{2^3}, 1 - \frac{1}{2^4}, \ldots, 1 - \frac{1}{2^{n-1}}, \ldots\right); \qquad \text{and}$$

$$x_i = (0, \ldots, 0, 1, 0, \ldots, 0, \ldots), \qquad i = 1, 2, \ldots, n-1$$

then it is easy to see that

$$P(a,b) \cap P(x_1, \ldots, x_{n-1}) = \{0\}, \qquad \|a, x_1, \ldots, x_{n-1}\| = \|b, x_1, \ldots, x_{n-1}\| = 1$$

$$\|a+b, x_1, x_2, \ldots, x_{n-1}\| = 2 = \|a, x_1, x_2, \ldots, x_{n-1}\| + \|b, x_1, x_2, \ldots, x_{n-1}\|,$$

but $a \neq b$. This implies that $(l^\infty, \|\bullet, \ldots, \bullet\|)$ is not strong convex.

3. In [3] was given the following definition of a strong n–convex n–normed space: we call the n–normed space $(L, \|\bullet, \ldots, \bullet\|)$ a strong n–convex if for every vectors $x_1, \ldots, x_{n+1} \in L$ which satisfies the conditions:

$$\|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_{n+1}\| = \frac{1}{n+1}\|x_1+x_{n+1}, x_2+x_{n+1}, \ldots, x_n+x_{n+1}\| = 1,$$

for $i = 1, 2, \ldots, n+1$ it is true that $x_{n+1} = \sum_{i=1}^{n} x_i$.

In the same work it is proved that every strong convex n–normed space is strong n–convex. The converse is not true. If $n = 2$ in [5] was given an example of a strong n–convex n–normed space which is not strong convex. It is naturally to ask does $(l^\infty, \|\bullet, \ldots, \bullet\|)$ is a strong n–convex n–normed space.

References

[1] Малчески, Р.: *Забелешки за n–нормирани простори*, Математички билтен **20** (1996).

[2] Malčeski, R.: *Strong n–convex n–normed spaces*, Matematički bilten **21**(1977).

[3] Malčeski, R.: *Strong convex n–normed spaces*, (preprint).

[4] Misiak, A.: *n–Inner Product Spaces*, Math. Nachr. **140** (1989).

[5] Diminnie, C., Gahler, S., White, A.: *Remarks on Strictly Convex and Strictly 2–Convex 2–Normed Spaces*, Math. Nachr. **88** (1979).

[6] Rudin, W.: *Functional Analysis*, 2d ed., McGraw–Hill Book Company, New York, (1991).

l^∞ КАКО n–НОРМИРАН ПРОСТОР

Алекса Малчески

Резиме

Концептот за n–норма на векторски простор со димензија поголема од n, $n \in N$ е воведен од А. Мисиак ([4]). Тоа е повеќедимензионална аналогија на поимот за норма.

Во [1], [2], [3] и [4] се докажани некои својства на n–нормираните простори. Во оваа работа е докажано дека во просторот од ограничени низи реални броеви со вообичаените операции собирање и множење со скалар е реален n–нормиран простор.

Faculti of enginearing

P.O. Box 162

91000 Skopje

Macedonia