Математички Билтен
23 (XLIX)
1999 (11–18)
Скопје, Македонија

ISSN 0351-336X

CANONICAL GROUPOIDS WITH $x^m \cdot y^n = xy$

Ǵ. Čupona*, N. Celakoski** and B. Janeva***

Abstract

We give a convinient description of free objects in the variety of groupoids with the axiom $x^m \cdot y^n = xy$.

1. Main Results

First we state necessary preliminaries.

Among all the possible $\dfrac{(2k-2)!}{k!(k-1)!}$ k-th groupoid powers, here x^k is defined as follows:

$$x^1 = x\,, \quad x^{k+1} = x^k \cdot x\,, \tag{1.1}$$

and this is the meaning of the powers in the axiom

$$x^m \cdot y^n = xy \tag{1.2}$$

of the variety of groupoids $\mathcal{U}^{(m,n)}$.

Throughout the paper, we assume that $\boldsymbol{F} = (F, \cdot)$ is an absolutely free groupoid with a given basis B. A mapping $x \mapsto P(x)$, from F into the family of finite nonempty subsets of F is defined as follows:

$$P(b) = \{b\}, \quad P(tu) = \{tu\} \cup P(t) \cup P(u)\,, \tag{1.3}$$

for any $b \in B$, t, $u \in F$. (We say that $P(u)$ is a *part* of u.)

We say that a groupoid $\boldsymbol{R} = (R, *)$ is a $\mathcal{U}^{(m,n)}$-*canonical groupoid* iff the following conditions hold:

a) $B \subseteq R \subseteq F$; b) $(\forall t,\, u \in F)\ (tu \in R \Rightarrow t,\, u \in R\ \&\ t * u = tu)$;

c) $\boldsymbol{R}$ is free in $\mathcal{U}^{(mn)}$ with the basis B.

The following statement is a special case of the main result in the paper [5, Theorem 1].

Theorem 1. *Assume that*

$$(m,n) \in \{(i,j) : (i = 1 \ \text{ or } \ j = 1)\ \&\ i,j \in \mathbb{N}\} \cup \{(2,2)\}\,.$$ [1]

[1] $\mathbb{N}$ is the set of positive integers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of nonnegative integers.

Then, there exist two transformations ξ, η on F such that the structure $\boldsymbol{R} = (R, \bullet)$ *defined by:*

$$B \subseteq R \;\&\; (\forall t,\, u \in R)(tu \in R \Leftrightarrow \xi(t) = t,\, \eta(u) = u)\,,$$
$$(\forall t,\, u \in R) t \bullet u = \xi(t)\, \eta(u)\,.$$

is a $\mathcal{U}^{(m,n)}$-canonical groupoid.

Now, we shall state the main results in this paper.

Theorem 2. *If $m, n \in \mathbb{N}$ are such that $m, n \geq 2$ and $m + n \geq 5$, then there exist mappings $\xi, n\colon F \times \mathbb{N}_0 \to F$ such that a $\mathcal{U}^{(m,n)}$-canonical groupoid* $\boldsymbol{R} = (R, \bullet)$ *is defined as follows:*

$$R = \{v \in F\colon (\forall t,\, u \in F, p, q \geq 0)\xi(t, p+1) \cdot \eta(u, q+1) \notin P(v)\}$$

$$(\forall v,\, w \in R) v \bullet w = \begin{cases} vw, & \text{if } vw \in R \\ \xi(t,p) \bullet \eta(u,q), & \text{if } vw = \xi(t, p+1) \cdot \eta(u, q+1)\,. \end{cases}$$

Theorem 3. *The class of free objects in $\mathcal{U}^{(m,n)}$ is hereditary iff $n = 1$ or $(m = 1,\ n = 2)$.*

2. Some Properties of F and $\mathcal{U}^{(m,n)}$

In the proof of Theorem 2 we shall use another three kinds of groupoid powers: $x^{(k,p)}$, $x^{\langle p\rangle}$, $x^{[p]}$. We define them below, assuming that k, n, m, p, q are integers such that $k \geq 1$, $n \geq 3$, $2 \leq m < n$, $p, q \geq 0$:

$$x^{(k,0)} = x,\ x^{(k,p+1)} = (x^{(k,p)})^k\,,$$
$$x^{\langle 0\rangle} = x,\ x^{\langle p+1\rangle} = x^2\, \underline{x^{\langle p\rangle} n-2}\,,$$
$$x^{[0]} = x,\ x^{[1]} = x^n,\ x^{[p+2]} = x^{[p+1]} x^{[p]}\, \underline{x^{[p+1]} n-m-1}\,,$$

where

$$xyz = (xy)z,\quad xy\,\underline{z0} = xy,\quad xy\,\underline{zp+1} = (xy\,\underline{zp})z\,.$$

It is clear that

$$(x^{(k,p)})^{(k,q)} = x^{(k,p+q)}\,,$$

and it is easy to show that the following identities hold in $\mathcal{U}^{(m,n)}$:

$$x^{(m,p)} y^{(n,p)} = xy, \qquad (\text{for } m, n \geq 1,\ p \geq 0)\,, \tag{2.1}$$
$$x^{\langle p\rangle} = x^{(n,p)}, \quad (\text{for } m = n \geq 3,\ p \geq 0)\,, \tag{2.2}$$
$$x^{[p]} = x^{(n,p)}, \quad (\text{for } 2 \leq m < n,\ p \geq 0)\,. \tag{2.3}$$

Some properties of the groupoid $\boldsymbol{F}$ shall be stated below. First, note that $\boldsymbol{F}$ is injective[2] groupoid, and that the basis B of $\boldsymbol{F}$ consists of the set of primes[3] in $\boldsymbol{F}$. As a consequence, we obtain that the mapping $x \mapsto P(x)$, is well defined by (1.3).

---

[2] A grupoid $\boldsymbol{G}$ is *injective* iff $(\forall x, y, u, v \in G)\,(xy = uv \Rightarrow x = u,\ y = v)$.

[3] An element $a \in G$ is *prime* in $\boldsymbol{G}$ iff $(\forall x,\, y \in G) a \neq xy$.

We denote by $|\,|: x \mapsto |x|$ the homomorphism from $\boldsymbol{F}$ into the additive groupoid of positive integers which is an extension of the mapping $B \to \{1\}$. Thus:

$$|b| = 1, \ |tu| = |t| + |u|, \tag{2.4}$$

for any $b \in B$, t, $u \in F$. (We say that $|t|$ is the *lenght* of t.)

By [5, (2.4)], the definitions of groupoid powers and (2.4) it follows that, for $i \geq 1$, $k \geq 2$, $n \geq 3$, $2 \leq m < n$, $p, q \geq 0$, the following relations hold in $\boldsymbol{F}$:

$$\begin{gathered} |x^{i+1}| > |x^i|, \ |x^{(k,p+1)}| > |x^{(k,p)}|, \\ |x^{\langle p+1\rangle}| > |x^{\langle p\rangle}|, \ |x^{[p+1]}| > |x^{[p]}|. \end{gathered} \tag{2.5}$$

By the injectivity of $\boldsymbol{F}$ and (2.5) we obtain the following properties:

$$t^{i+1} = u^{j+1} \Rightarrow i = j, \ t = u; \tag{2.6}$$

$$t^{(k,p+q)} = u^{(k,p)} \Rightarrow u = t^{(k,q)}; \tag{2.7}$$

$$t^{(k+i,p)} = u^{(k,q)} \Rightarrow p = q = 0, \ t = u; \tag{2.8}$$

$$t^{\langle p+1\rangle} = u^{\langle q+1\rangle} \Rightarrow t = u, \ p = q; \tag{2.9}$$

$$t^{[p+1]} = u^{[q+1]} \Rightarrow t = u, \ p = q; \tag{2.10}$$

$$t^{(m,p)} = u^{[q]} \Rightarrow p = q = 0; \tag{2.11}$$

where $i, j \geq 1$, $k \geq 2$, $p, q \geq 0$ $t, u \in F$.

By (2.7) it follows that for a given $u \in F$ and $k \geq 1$, there exists at most one pair (t, p), such that $u = t^{(k,p+1)}$. If such a pair (t,p) exists, then we write $(u)_k = p$, $t = u^{(k,-)}$, and if $(u)_k = 0$, then $t = u = u^{(k,-)}$. If k is fixed, we shall often write $x^{(p)}$, (x), $x^{(-)}$ instead of $x^{(k,p)}$, $(x)_k$, $x^{(k,-)}$-respectively. In the same sence, by (2.9) and (2.10), for a given $u \in F$ one defines $\langle u\rangle$, $[u]$-respectively.

3. Proofs of Theorems

As we mentioned in Section 1, Theorem 1 is a special case of Theorem 1 in [5]. Therefore we shall merely define the corresponding transformations ξ and η, without entering the proof in details.

Case $\underline{m = n = 1}$. $\xi(u) = \eta(u) = u$, for every $u \in F$,

Therefore $\boldsymbol{F}$ is the $\mathcal{U}^{(1,1)}$-canonical groupoid.

Case $\underline{m = 1, \ n \geq 2}$. $\xi(u) = u$, for every $u \in F$,

$$\eta(u) = \begin{cases} u, & \text{if } u \neq t^k, \ t \in F, \ k \geq 2, \\ t, & \text{if } u = t^k, \ k \geq 2, \ t \in F. \end{cases}$$

Case $\underline{m \geq 2, \ n = 1}$. First, for $x \in F$, $p \geq 0$, define $f(x,p)$ by:

$$f(x,0) = x, \ \ f(x,p+1) = x\,\underline{f(x,p)m}.$$

Then: $\eta(u) = u$, for every $u \in F$, and

$$\xi(u) = \begin{cases} u\,, & \text{if } (\forall t \in F\,, p \geq 0) u \neq f(t, p+1)\,, \\ t\,, & \text{if } u = f(t, p+1)\,. \end{cases}$$

Case $\underline{m = n = 2}$.

$$\xi(u) = \eta(u) = \begin{cases} u\,, & \text{if } (\forall t \in F) u \neq t^2\,, \\ t\,, & \text{if } u = t^2\,. \end{cases}$$

We begin the proof of Theorem 1, assuming that $m \geq 2$, $n \geq 2$, $m+n \geq 5$. First of all, the identity (2.1) suggests to define ξ and η by:

$$\xi(t,p) = t^{(m,p)}\,, \quad \eta(t,p) = t^{(n,p)}\,, \tag{3.1}$$

having in mind the definition of $x^{(k,p)}$, given in Section 2.

The next proposition implies that the definition (3.1) is "successful" only for $m > n \geq 2$.

Proposition 3.1. *Let* $m+n \geq 5$, $m \geq 2$, $n \geq 2$, *and* $R_{(m,n)}(= R)$ *be the subset of* F *defined by:*

$$R = \{v \in F\colon (\forall t,\, u \in F)\big((t)_m(u)_n \neq 0 \Rightarrow tu \notin P(v)\big)\}\,. \tag{3.2}$$

For v, $w \in R$, *let* $v \bullet w$ *be defined by:*

$$v \bullet w = \begin{cases} vw\,, & \text{if } vw \in R\,, \\ t^{(m,p)} \bullet u^{(n,q)}\,, & \text{if } v = t^{(m,p+1)},\ w = u^{(n,q+1)}\,. \end{cases} \tag{3.3}$$

Then

(i) $\boldsymbol{R} = (R, \bullet)$ *is a grupoid,* B *coincides with the set of primes in* $\boldsymbol{R}$, *and it is the least generating set for* $\boldsymbol{R}$.

(ii) *If* $\boldsymbol{G} \in \mathcal{U}^{(m,n)}$, *and* $\lambda\colon B \to G$ *is a mapping, then there exists a unique homomorphism* $\varphi\colon \boldsymbol{R} \to \boldsymbol{G}$ *which extends* λ.

(iii) $\boldsymbol{R} \in \mathcal{U}^{(m,n)}$ *iff* $m > n \geq 2$.

(iv) *If* $m > n \geq 2$, *then* $\boldsymbol{R}$ *is* $\mathcal{U}^{(m,n)}$*-free groupoid with a (unique) basis* B.

Proof. (i) From (3.2), (3.3) and (2.7) it follows that $\bullet$ is a well-defined operation on R. The assertion for B follows from the fact that B is the set of primes in $\boldsymbol{F}$ and it is the least generating set for $\boldsymbol{F}$.

(ii) Let $\psi\colon \boldsymbol{F} \to \boldsymbol{G}$ be the homomorphism which extends λ, and let $\varphi\colon R \to G$ be the restriction of ψ on R. By (3.3), $\boldsymbol{G} \in \mathcal{U}^{(m,n)}$ and (2.1), it follows that $\varphi\colon \boldsymbol{R} \to \boldsymbol{G}$ is a homomorphism.

(iii) If $t \in R$, $i \geq 1$, then we denote by $t^i_\bullet$ the i-th power of t in $\boldsymbol{R}$, i.e. $t^1_\bullet = t$, $t^{i+1}_\bullet = (t^i_\bullet) \bullet t$.

($iii.1$) Let $m > n \geq 2$. Then: $t^i_\bullet = t^i$, for each $i\colon 1 \leq i \leq m$. (If $a \in B$, then $a^i_\bullet = a^i$, for every $i \geq 1$.) Hence, for t, $u \in R$, we have:

$$(t^m_\bullet) \bullet (u^n_\bullet) = t^m \bullet u^n = t^{(m,1)} \bullet u^{(n,1)} = t^{(m,0)} \bullet u^{(m,0)} = t \bullet u\,,$$

i.e. $\boldsymbol{R} \in \mathcal{U}^{(m,n)}$.

($iii.2$) If $2 \leq m = n$, and $a \in B$, then we have:

$$\left((a^n)^n_\bullet\right) \bullet a^n = (a^2\, \underline{a^n n-2}) \bullet a^n = a^2\, \underline{a^n n - 1} \neq a^{n+1}\,,$$

and thus, $\boldsymbol{R} \notin \mathcal{U}^{(m,n)}$.

(*iii*.3) Let $2 \leq m < n$, and $a \in B$. Then $(a^n)^n_\bullet = a^{n+1}\, \underline{a^n n - m - 1}$, and therefore

$$(a^m_\bullet) \bullet (a^n)^n_\bullet = a^m(a^{n+1}\, \underline{a^n n - m - 1}) \neq a \bullet a^n = aa^n\,.$$

This completes the proof of part (*iii*).

As a consequence of (*i*), (*ii*) and (*iii*) one obtains that (*iv*) is true. □

The following property proves Theorem 2 in the case $m = n \geq 3$.

Proposition 3.2. *Let $n \geq 3$ and let $S_n(= S)$ be the subset of F defined by:*

$$S = \{v \in F\colon (\forall t,\ u \in F)\big(\langle t\rangle\langle u\rangle > 0 \Rightarrow tu \notin P(v)\big)\,. \tag{3.4}$$

For $v, w \in S$ define $v \bullet w$ by:

$$v \bullet w = \begin{cases} vw\,, & \text{if } vw \in S \\ t^{\langle p\rangle} \bullet u^{\langle q\rangle}\,, & \text{if } vw = t^{\langle p+1\rangle} u^{\langle q+1\rangle}\,. \end{cases} \tag{3.5}$$

Then $\boldsymbol{S} = (S, \bullet)$ is a $\mathcal{U}^{(n,n)}$-free groupoid with the unique basis B.

Proof.

1) Since $|t^{\langle p+1\rangle}| > |t^{\langle p\rangle}|$, we obtain $\boldsymbol{S} = (S, \bullet)$ is a well-defined groupoid. It is clear that B coincides with the set of primes in $\boldsymbol{S}$ and it generates $\boldsymbol{S}$.

2) Now we shall prove that $\boldsymbol{S} \in \mathcal{U}^{(n,n)}$.

Let $v,\ w \in S$ be such that $0 \leq p \leq \langle v\rangle, 0 \leq q \leq \langle w\rangle, v = t^{\langle p\rangle},\ w = u^{\langle q\rangle}$. Then

$$v^2_\bullet = t^2\,,\ v^3_\bullet = t^2 t^{\langle p\rangle}, \dots, v^n_\bullet = t^2\, \underline{t^{\langle p\rangle} n - 2} = t^{\langle p+1\rangle}\,.$$

In the same way we obtain that: $w^n_\bullet = u^{\langle q+1\rangle}$. Therefore:

$$(v^n_\bullet) \bullet (w^n_\bullet) = t^{\langle p+1\rangle} \bullet u^{\langle q+1\rangle} = t^{\langle p\rangle} \bullet u^{\langle q\rangle} = v \bullet w\,,$$

i.e. we obtain that $\boldsymbol{S} \in \mathcal{U}^{(n,n)}$. [4]

3) Let $\boldsymbol{G} = (G, \cdot) \in \mathcal{U}^{(n,n)}$, $\lambda\colon B \to G$ be a given mapping and $\psi\colon \boldsymbol{F} \to \boldsymbol{G}$ the homomorphism which extends λ. Then (using the fact that the identity (2.2), i.e. $x^{(p)} = x^{\langle p\rangle}$ holds in $\boldsymbol{G}$) we obtain that the restriction φ of ψ on S, i.e. $\varphi\colon \boldsymbol{S} \to \boldsymbol{G}$, is a homomorphism.

From 1), 2) and 3) it follows that $\boldsymbol{S}$ is a $\mathcal{U}^{(n,n)}$-free groupoid with the basis B, and that B coincides with the set of primes in $\boldsymbol{S}$. □

It remains the case $2 \leq m < n$. Bellow we shall write $x^{(p)}$, (x) instead od $x^{(m,p)}$, $(x)_m$-respectively.

---

[4] We note that, if $\boldsymbol{G} \in \mathcal{U}^{(2,2)}$, then the identity $x^2 = x^k$ is true in $\boldsymbol{G}$ for every $k \geq 2$ and every groupoid k-th power among all possible groupoid k-th powers.

Proposition 3.3. *Let $2 \leq m < n$ and let the structure $\boldsymbol{T} = (T, \bullet)$ be defined as follows:*

$$T = \{v \in F \colon (\forall t,\, u \in F, p, q \geq 0)\big((t)[u] \neq 0 \Rightarrow tu \notin P(v)\big)\,,$$

and for $v, w \in F$:

$$v \bullet w = \begin{cases} vw\,, & \text{if } vw \in T \\ t^{(p)} \bullet u^{[q]}\,, & \text{if } v = t^{(p+1)},\ w = u^{[q+1]}\,. \end{cases}$$

Then $\boldsymbol{T}$ is a $\mathcal{U}^{(m,n)}$-free groupoid with the unique basis B, where B coincides with the set of primes in $\boldsymbol{T}$.

Proof.

1) By the same reasoning as for the proof of the Proposition 3.2, $\boldsymbol{T}$ is a well defined groupoid, B is the least generating set of $\boldsymbol{T}$, and it coincides with the set of primes in $\boldsymbol{T}$.
2) By (2.11), $(v)[v] = 0$ for every $v \in F$, and this implies that $v_\bullet^i = v^i$, for every $v \in T$ and $1 \leq i \leq m$. Moreover, for $[v] = 0$, we have:
$$v_\bullet^{m+1} = v^m \bullet v = v^{m+1}, \ldots, v_\bullet^n = v^n\,.$$
In the case $[v] = p + 1$, $v = t^{[p+1]}$, we have:
$$v_\bullet^{m+1} = v^m \bullet t^{[p+1]} = vt^{[p]} = t^{[p+1]}t^{[p]}\,,$$
$$v_\bullet^n = t^{[p+1]}t^{[p]}\,\underline{t^{[p+1]}n - m - 1} = t^{[p+2]}\,.$$
Therefore:
$$(v_\bullet^m) \bullet (w_\bullet^n) = v^m \bullet u^{[q+1]} = v \bullet u^{[q]} = v \bullet w\,,$$
where $[w] = q$, $w = u^{[q]}$.
This shows that $\boldsymbol{T} \in \mathcal{U}^{(m,n)}$.
3) In the same way as in the proof of Proposition 3.2 one shows that $\boldsymbol{T}$ is $\mathcal{U}^{(m,n)}$-free with the (unique) basis B. □

Proposition 3.1-3.3 complete the proof of Theorem 2. Here, we define ξ and η by:

$$\xi(x,p) = x^{(m,p)},\quad \eta(x,p) = x^{(n,p)},\qquad \text{for } m > n \geq 2\,,$$

$$\xi(x,p) = \eta(x,p) = x^{(p)},\quad \text{for } m = n \geq 3\,,$$

$$\xi(x,p) = x^{(m,p)},\quad \eta(x,p) = x^{[p]},\quad \text{for } 2 \leq m < n\,.$$

It remains to prove Theorem 3. We note that the part of the assertion in Theorem 3 for $(m,n) \in \{(i,j) \colon i = 1 \text{ or } j = 1\} \cup \{(2,2)\}$ is a special case of [5, Theorem 2], and therefore we shall prove the following:

Proposition 3.4. *If $m, n \geq 2$, $m+n \geq 5$, then the class of $\mathcal{U}^{(m,n)}$-free groupoids is not hereditary.*

Using the fact that, if $\boldsymbol{Q}$ is a free object in $\mathcal{U}^{(m,n)}$ then the set of primes in $\boldsymbol{Q}$ is the basis of $\boldsymbol{Q}$, we obtain that Proposition 3.4 is a consequence of the following:

Proposition 3.5. *Let $m \geq 2$, $n \geq 2$, $\boldsymbol{G} \in \mathcal{U}^{(m,n)}$ and $a \in G$. If $\boldsymbol{Q}$ is the subgroupoid of $\boldsymbol{G}$ generated by $\{a^m, a^n\}$, then there are no primes in $\boldsymbol{Q}$.*

Proof. Let us first define a sequence $\{f_i : i \geq 2\}$ of transformations on G. Namely,

$$f_2(x) = a^m a^n\,, \quad f_{k+1}(x) = \big(f_k(x)\big)^m x^n\,.$$

Then, $a^k = f_k(a) = \big(f_{k-1}(a)\big)^m a^n$ is not prime for any $k \geq 2$. Thus, neither a^m, nor a^n is a prime in $\boldsymbol{Q}$. Therefore there are no primes in $\boldsymbol{Q}$. □

4. Relations between $\mathcal{U}^{(m,n)}$, $\mathcal{U}^{(m,1)}$, $\mathcal{U}^{(1,n)}$

By the definition of the variety $\mathcal{U}^{(m,n)}$ it is clear that

$$\mathcal{U}^{(m,1)} \cap \mathcal{U}^{(1,n)} \subseteq \mathcal{U}^{(m,n)}\,. \tag{4.1}$$

It is natural to seak for the answer of the question: what are the cases for which equality holds in (4.1)? The answer is given in the following.

Proposition 4.1. *The equality holds in* (4.1) *in the following three cases only:*

$$a)\ m = 1, \quad b)\ n = 1, \quad c)\ m = n = 2.$$

Proof. It is clear that the equality holds in (4.1) for each of the cases a), b), and it is easily shown that the equality holds in the case c), as well (see, for example, [3, 1.1 and Remark]). In any other case, the equality $\mathcal{U}^{(m,1)} \cap \mathcal{U}^{(1,n)} = \mathcal{U}^{(m,1)} \cap \mathcal{U}^{(1,d+1)}$ holds, where $d = \gcd(m-1, n-1)$ (the greatest common divisor of $m-1$ and $n-1$). In this case it is easy to show that $\mathcal{U}^{(m,n)}$-canonical groupoid does not belong to the variety $\mathcal{U}^{(m,1)}$. □

References

[1] R. H. Bruck: *A Survey of Binary Systems*, Berlin-Göttingen-Heidelberg, 1956

[2] P. M. Cohn: *Universal Algebra*, Harpers & Row Publ., 1965

[3] Ǵ. Čupona, N. Celakoski: *On Groupoids with the identity $x^2y^2 = xy$*, Contributions, Sec. Math. Tech. Sci., MANU (in print).

[4] Ǵ. Čupona, N. Celakoski: *Free groupoids with $xy^2 = xy$*, Bulletin Mathématique, Tome 21 (1997), 5-16

[5] Ǵ. Čupona, N. Celakoski, B. Janeva: *Free Groupoids with the Axioms of the Form $x^{m+1} \cdot y = xy$ and/or $x \cdot y^{n+1} = xy$* (unpublished).

[6] Ǵ. Čupona, N. Celakoski, B. Janeva: *Varieties of Groupoids with the Axioms of the Form $x^{m+1} \cdot y = xy$ and/or $x \cdot y^{n+1} = xy$* (unpublished).

[7] S. Markovski: *Finite Mathematics*, Skopje, 1993 (in Macedonian).

КАНОНИЧНИ ГРУПОИДИ СО $x^m y^n = xy$

Ѓ. Чупона*, Н. Целакоски** и Б. Јанева***

Резиме

Во овој труд се разгледуваат многуобразијата групоиди, определени со аксиома од обликот $x^m y^n = xy$, кои што се означени со $\mathcal{U}^{(m,n)}$. Добиен е погоден опис на слободен групоид со база B, а имено групоид $\mathbf{R} = (R, \star)$ којшто ги задоволува следниве услови:

а) $B \subseteq R \subseteq F$; б) $(\forall t, u \in F)(tu \in R \Rightarrow t, u \in R \& t \star u = tu)$;

в) $\mathbf{R}$ е слободен во $\mathcal{U}^{(m,n)}$ со база B;

$\mathbf{R}$ се нарекува $\mathcal{U}^{(m,n)}$-каноничен групоид. Со $\mathbf{F} = (F, \cdot)$ се означува апсолутно слободниот групоид со база B.

Докажани се следниве теореми:

Теорема 1. *Нека*

$$(m,n) \in \{(i,j) : (i = 1 \text{ or } j = 1) \& i, j \in \mathbf{N}\} \cup \{(2,2)\}.$$

Тогаш постојат две трансформации ξ, η на F такви што $\mathbf{R} = (R, \bullet)$ определен со:

$$B \subseteq R \ \& \ (\forall t, u \in R)(tu \in R \iff \xi(t) = t, \ \eta(u) = u),$$
$$(\forall t, u \in R) t \bullet u = \xi(t)\, \eta(u).$$

е $\mathcal{U}^{(m,n)}$-каноничен групоид.

Теорема 2. *Ако $m, n \in \mathbf{N}$ се такви што $m, n \geq 2$ и $m + n \geq 5$, тогаш постојат пресликувања $\xi, \eta : F \times \mathbf{N}_0 \to F$ такви што $\mathcal{U}^{(m,n)}$-каноничниот групоид $\mathbf{R} = (R, \bullet)$ е дефиниран со:*

$$R = \{v \in F \colon (\forall t, u \in F, p, q \geq 0) \xi(t, p+1) \cdot \eta(u, q+1) \notin P(v)\}$$

$$(\forall v, w \in R) v \bullet w = \begin{cases} vw, & \text{ако } vw \in R \\ \xi(t,p) \bullet \eta(u,q), & \text{ако } vw = \xi(t, p+1) \cdot \eta(u, q+1). \end{cases}$$

Покрај тоа дадена е карактеризација на многуобразијата $\mathcal{U}^{(m,n)}$, такви што класата слободни објекти е наследна. Имено докажана е:

Теорема 3. *Класата слободни објекти во $\mathcal{U}^{(m,n)}$ е наследна акко $n = 1$ или $(m = 1, n = 2)$.*

На крајот, покажано е дека равенството

$$\mathcal{U}^{(m,1)} \cap \mathcal{U}^{(1,n)} = \mathcal{U}^{(m,n)}$$

важи акко $m = 1$ или $n = 1$ или $m = n = 2$.

* Macedonian Academy od Sciences and Arts, Skopje, Macedonia

** Faculty of Mechanical Engineering, Skopje, Macedonia

*** Faculty of Natural Sciences and Mathematics, Skopje, Macedonia