Математички Билтен ISSN 0351-336X
33(LIX)
2009 (81-90)
Скопје, Македонија

DETECTION FUNCTION METHOD AND ITS APPLICATION TO A PERTURBED HAMILTONIAN SYSTEM

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Abstract. A Hamiltonian system with 25 singular points under five-order perturbed terms is introduced in this paper in order to study the existence, number and distribution of limit cycles. The detection function method is employed. First we classify the type of the closed curves of the unperturbed system, then we obtain the analytical forms of the detection functions of the perturbed system. Finally, by numerical explorations, we point out the distribution of the limit cycles for a particular case.

1. Introduction and preliminary results

Recently, studies on the properties of the limit cycles are becoming of considerable interest not only for mathematicians but also for physicists. Even more, recently chemists, biologists, economists, started working intensively on this subject [15],[14],[6]. When dealing with the existence of limit cycles of a perturbed Hamiltonian system, it is assumed that the corresponding unperturbed Hamiltonian system possesses at least a center. A center is an isolated singular point surrounded by a continuous family of periodic orbits. A limit cycle [14] is an isolated periodic orbit. The problem of studying the existence and number of limit cycles for a system of polynomial differential equations of a given degree is known as belonging to the 16-th Hilbert problem. It is known that this problem is open even for planar quadratic polynomial systems. There are examples of such systems which have at least four limit cycles but, as long as we know, no proof unanimously accepted exists for the fact that any planar quadratic polynomial systems can have *at most* four limit cycles. A recent challenging proof of this fact has been put forward in [4] but other researchers expressed doubts to it. The fact that any polynomial vector field has finitely many limit cycles, remains maybe the best result on this topic [7], [3].

The reason of the present work is to try to find some relationships between the polynomial's degree and the number of limit cycles in a particular case. Can a polynomial system of a relative small degree have a large number of limit cycles? As in the literature, as long as we know, there is no method for attacking the problem in its whole generality, we choose here to investigate a particular system. It is known that many times results obtained for particular cases give an intuitive

Key words and phrases. Limit cycles, Abelian integral, Hamiltonian systems.
M.S.C. 2000: 34C07, 37G15.

conjecture for the general case. Many other particular systems are proposed on this topic in a sustained effort to propose a conjecture and ultimately to prove it.

In the present work, we apply the Abelian integral [1] in order to find the number and distribution of the limit cycles for a given perturbed Hamiltonian system.

The system of differential equations:

$$\begin{cases} \dot{x} = y\left(1 + x^2 - dy^2\right) + \varepsilon x(mx^2 + ny^2 - \lambda) \\ \dot{y} = -x\left(1 - cx^2 + y^2\right) + \varepsilon y(mx^2 + ny^2 - \lambda) \end{cases} \tag{1}$$

has been discussed in [9] and the system

$$\begin{cases} \dot{x} = 4y\left(abx^2 - by^2 + 1\right) + \varepsilon x(ux^n + vy^n - b\frac{\beta+1}{\mu+1}x^\mu y^\beta - ux^2 - \lambda) \\ \dot{y} = 4x\left(ax^2 - aby^2 - 1\right) + \varepsilon y(ux^n + vy^n + bx^\mu y^\beta - vy^2 - \lambda) \end{cases} \tag{2}$$

in [12], [13] where $cd > 1, d > c > 0,\ \mu + \beta = n, 0 < a < b < 1, 0 < \varepsilon \ll 1$, u, v, m, n, λ are the real parameters of the systems and $n = 2k,\ k \neq 0$ a natural number.

In the present work we consider a system, invariant under a rotation of π, given in polar coordinates by:

$$\begin{aligned} &\dot{r} = r^5\left[a_4\left(\cos 6\theta + \cos 2\theta\right) + a_5(\sin 6\theta + \sin 2\theta)\right] \\ &\dot{\theta} = a_1 + a_2 r^2 + a_3 r^4 + r^4\left[a_5\left(\cos 6\theta + 3\cos 2\theta\right) - a_4\left(\sin 6\theta + 3\sin 2\theta\right)\right]. \end{aligned} \tag{3}$$

The system is a Hamiltonian one [8] with the Hamiltonian function

$$\begin{aligned} H(r,\theta) = -\frac{1}{2}a_1 r^2 - \frac{1}{4}a_2 r^4 - \frac{1}{6}a_3 r^6 - \frac{1}{6}r^6\left(a_5\cos 6\theta - a_4\sin 6\theta\right) \\ - \frac{1}{2}r^6\left(a_5\cos 2\theta - a_4\sin 2\theta\right), \end{aligned} \tag{4}$$

with $a_i, i = 1 - 5$ real numbers. In cartesian coordinates (4) reads:

$$\begin{aligned} H_1(x,y) = -\frac{1}{2}a_1(x^2+y^2) - \frac{1}{4}a_2(x^2+y^2)^2 - \frac{1}{6}a_3(x^2+y^2)^3 \\ + \frac{2}{3}a_4 xy(3x^4 - 2x^2y^2 + 3y^4) - \frac{2}{3}a_5(x^2-y^2)(x^4 - 2x^2y^2 + y^4). \end{aligned}$$

The equilibrium points lie on the lines:

$d_1 : \theta = \frac{\pi}{4} + k\pi; d_2 : \theta = -\frac{\pi}{4} + k\pi, k = 0, 1;$

$d_3 : \theta = (-\frac{\pi}{4} - 2k\pi)/4; d_4 : \theta = (\frac{3\pi}{4} - 2k\pi)/4, k = 0, 2;$

$d_5 : \theta = (-\frac{\pi}{4} - 2k\pi)/4; d_6 : \theta = (\frac{3\pi}{4} - 2k\pi)/4, k = 1, 3.$

The system (3) could have at most 25 equilibrium points. After some computations we observe that the system (3) has exactly 25 equilibrium points if the parameters fulfil the conditions: $a_2^2 - 4a_1b_i > 0,\ a_2b_i < 0,\ a_1b_i > 0, i = 1, 2, 3$, where

$b_1 = a_3 \pm 2, \quad b_2 = a_3 \pm 2\sqrt{2(a_4^2+b_4^2)}(\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}),$
$b_3 = a_3 \pm 2\sqrt{2(a_4^2+b_4^2)}(\sin\frac{\alpha}{2} - \cos\frac{\alpha}{2}),\ \cos\alpha = \frac{a_4}{\sqrt{a_4^2+b_4^2}}, \sin\alpha = \frac{a_5}{\sqrt{a_4^2+b_4^2}}$ and these points are:
$S_1^k\left(\frac{\pi}{4}+k\pi, r_1^+\right); S_2^k\left(\frac{\pi}{4}+k\pi, r_2^+\right); S_3^k\left(-\frac{\pi}{4}+k\pi, r_1^+\right); S_4^k\left(-\frac{\pi}{4}+k\pi, r_2^+\right), k=0,1;$
$S_5^k\left((-\frac{\pi}{4}+2k\pi)/4, r_1^+\right); S_6^k\left((-\frac{\pi}{4}+2k\pi)/4, r_2^+\right), k=0,2;$
$S_7^k\left((\frac{3\pi}{4}+2k\pi)/4, r_1^+\right); S_8^k\left((\frac{3\pi}{4}+2k\pi)/4, r_2^+\right), k=0,2;$
$S_9^k\left((-\frac{\pi}{4}+2k\pi)/4, r_1^+\right); S_{10}^k\left((-\frac{\pi}{4}+2k\pi)/4, r_2^+\right), k=1,3;$
$S_{11}^k\left((\frac{3\pi}{4}+2k\pi)/4, r_1^+\right); S_{12}^k\left((\frac{3\pi}{4}+2k\pi)/4, r_2^+\right), k=1,3,$
where r_1^+, r_2^+ are the corresponding positive solutions of the equation $\dot{\theta} = 0$.

This paper is organized as follows. In Section 2, we study the global portrait of the unperturbed Hamiltonian system, illustrating the shape of the closed curves (the level curves) of the form $H(r,\theta) = h$ as h varies on the real line. In Section 3, we find the analytical forms of the detection functions of the perturbed system. Finally, we point out the distribution of the limit cycles for a particular case.

2. Analysis of the unperturbed system

One way to produce limit cycles is by perturbing a Hamiltonian system which has one or more centers, in such a way that limit cycles bifurcate in the perturbed system from some of the periodic orbits in the original system. We have the following [2] results.

Theorem 2.1. *Consider the perturbed Hamiltonian system (5)*

$$\dot{x} = -\frac{\partial H}{\partial y} + P(x,y,\alpha), \dot{y} = \frac{\partial H}{\partial x} + Q(x,y,\alpha). \tag{5}$$

Assume that P, Q are polynomials with $P(x,y,0) = Q(x,y,0) = 0$, the curve C^h defined by Hamiltonian $H(x,y) = h$ of system (5) is a periodic orbit extending outside as h increases and $C^h(D)$ is the area inside C^h. If there exists h_0 such that function

$$A(h) = \iint\limits_{C^h(D)} [P''_{x\alpha}(x,y,0) + Q''_{y\alpha}(x,y,0)]dxdy \tag{6}$$

satisfies $A(h_0) = 0, A'(h_0) \neq 0, \alpha A'(h_0) < 0 (> 0)$, then system (5) has only one stable (unstable) limit cycle nearby C^{h_0} for α very small. If the C^h constricts inside as h increases, the stability of the limit cycle is opposite with above. If $A(h) \neq 0$, then system (5) has no limit cycles.

The integral $A(h)$ is called the Abelian integral and the problem is known as the weakened 16-th Hilbert problem.

Consider the system (3) with $a_1 = -2; a_2 = 12; a_3 = -6; a_4 = 1; b_4 = 1$. Then the system has 25 equilibrium points and they are:
$O(0,0); S_1^k\left(\frac{\pi}{4}+k\pi, 0.43702\right); S_2^k\left(\frac{\pi}{4}+k\pi, 1.144\,1\right), k=0,1;$
$S_3^k\left(-\frac{\pi}{4}+k\pi, 0.42086\right); S_4^k\left(-\frac{\pi}{4}+k\pi, 1.680\,1\right), k=0,1;$

$S_5^k\left((-\frac{\pi}{4}+2k\pi)/4, 0.41048\right); S_6^k\left((-\frac{\pi}{4}+2k\pi)/4, 3.9167\right), k=0,2;$
$S_7^k\left((\frac{3\pi}{4}+2k\pi)/4, 0.45453\right); S_8^k\left((\frac{3\pi}{4}+2k\pi)/4, 0.92862\right), k=0,2;$
$S_9^k\left((-\frac{\pi}{4}-2k\pi)/4, 0.43779\right); S_{10}^k\left((-\frac{\pi}{4}-2k\pi)/4, 1.1305\right), k=1,3;$
$S_{11}^k\left((\frac{3\pi}{4}-2k\pi)/4, 0.42028\right); S_{12}^k\left((\frac{3\pi}{4}-2k\pi)/4, 1.7182\right), k=1,3;$
and the values of the Hamiltonian $H(r,\theta)$ at each critical point, increasingly ordered, are:
$H\left(S_6^k\right)=-225.1; H\left(S_{12}^k\right)=-6.7477$; $H\left(S_4^k\right)=-6.0867$; $H\left(S_2^k\right)=-0.84085$; $H\left(S_{10}^k\right)=-0.78141$; $H\left(S_8^k\right)=-0.16872$; $H(O)=0; H\left(S_5^k\right)=0.08394$; $H\left(S_{11}^k\right)=8.6558\times10^{-2}$; $H\left(S_3^k\right)=8.6710\times10^{-2}$; $H\left(S_9^k\right)=9.1040\times10^{-2}$; $H\left(S_1^k\right)=9.0847\times10^{-2}$; $H\left(S_7^k\right)=9.5049\times10^{-2}$.
Evaluating the Jacobian matrix at each singular point we obtain that $O(0,0)$; S_2^k; $S_3^k, k=0,1$; S_6^k; $S_7^k, k=0,2$; S_9^k; $S_{12}^k, k=1,3$ are equilibrium points of type center and the rest are hyperbolic equilibrium points.

Theorem 2.2. *Numerical investigations reveal that, as h varies on the real line, the closed curves (level curves) defined by $H(r,\theta)=h$ can be divided into the following types:*

1. $L_1^h: -225.1\le h\le -6.7477$, this corresponds to two closed symmetric orbits, one orbit encircles S_2^0 and the other one encircles S_2^1, that is the second points on the line d_3, Fig. 1a) .

2. $L_2^h\cup L_1^h: -6.7477<h<-6.0867$, this corresponds to four closed symmetric orbits, one orbit encircles S_{12}^1 another one encircles S_{12}^3, Fig. 1a) (small orbits) and the other two are of type L_1^h.

3. $L_3^h: -6.0867\le h\le -0.84085$, this corresponds to two closed symmetric orbits which encircle respectively the second points on the lines d_2, d_3 and d_6, Fig. 1 b).

4. $L_4^h\cup L_3^h: -0.84085<h\le -0.78141$, this corresponds to two closed symmetric orbits which encircle respectively the second points on the line d_1 and another two of type L_3^h, Fig. 1 b) (small orbits).

5. $L_5^h: -0.78141<h<-0.16872$, this corresponds to two closed symmetric orbits which encircle the second point on the lines d_1, d_2, d_3, d_4, d_6, Fig. 2a).

6. $L_7^h\cup L_6^h: -0.16872\le h<0$, this corresponds to two closed orbits, one of which encircles all critical points and the other one encircles all first critical points on the lines $d_i, i=1-6$. If $h=0$ we get in addition the origin, Fig. 2b) .

7. $L_8^h\cup L_7^h\cup L_6^h: 0<h\le 0.08394$, this corresponds to a closed orbit that encircles only the origin and others two orbits of type L_7^h respectively L_6^h, Fig.2b).

8. $L_9^h\cup L_7^h: 0.08394<h<0.086558$, this corresponds to two closed symmetric orbits which encircles the first points on the lines d_1, d_2, d_4, d_5, d_6 and an orbit of type L_7^h (the orbit that surrounds all equilibria), Fig.3a) .

9. $L_{11}^h\cup L_{10}^h\cup L_7^h: 0.086558\le h<0.08671$, this corresponds to two closed symmetric (L_{11}^h) (small orbits) orbits which encircles the first points on the line d_2, two closed symmetric orbits (L_{10}^h) which encircle the first points on the lines d_1, d_4, d_5 and an orbit of type L_7^h, Fig. 3 b).

10. $L_{10}^h\cup L_7^h: 0.08671\le h\le 0.090847$, this corresponds to an orbit of type L_{10}^h and an orbit of type L_7^h, Fig. 3 b).

11. $L_{13}^h \cup L_{12}^h \cup L_7^h$: $0.090847 < h < 0.09104$, *this corresponds to two closed symmetric* (L_{13}^h) *orbits which encircle the first points on the line* d_5, *two closed symmetric orbits* (L_{12}^h) *(small orbits) which encircle the first points on the line* d_4 *and an orbit of type* L_7^h, *Fig. 3c).*

12. $L_{13}^h \cup L_7^h$: $0.09104 \leq h \leq 0.095049$, *this corresponds to two orbits of type* L_{13}^h *and to an orbit of type* L_7^h, *Fig. 3c).*

13. L_7^h : $h > 0.095049$, *this corresponds to an orbit of type* L_7^h.

As h *increases, the orbits* $L_1^h, L_2^h, L_3^h, L_4^h, L_5^h, L_7^h, L_8^h$ *expand towards outside, while the others orbits* $L_6^h, L_9^h, L_{10}^h, L_{11}^h, L_{12}^h, L_{13}^h$ *shrink inside.*

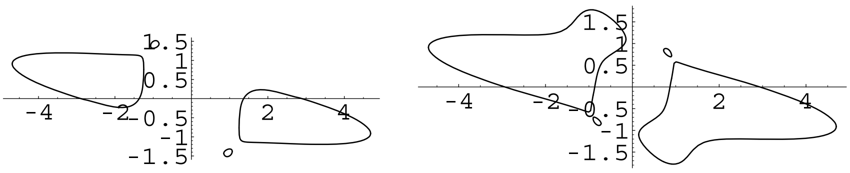

FIGURE 1. Orbits of type a) L_1, L_2 (left) b) L_3, L_4 (right)

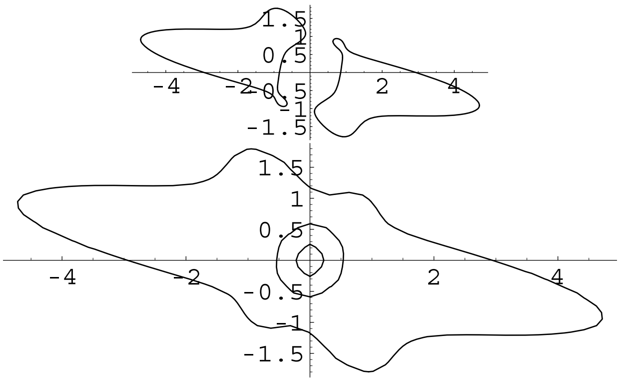

FIGURE 2. Orbits of type a) L_5 (up) b) $L_{6,7,8}$ (down)

3. THE DETECTION FUNCTION METHOD

Consider the following perturbed system [10]:

$$\begin{aligned} \dot{x} &= -\tfrac{\partial H}{\partial y} + \varepsilon x(p(x,y) - \lambda), \\ \dot{y} &= \tfrac{\partial H}{\partial x} + \varepsilon y(q(x,y) - \lambda), \end{aligned} \tag{7}$$

where $p(0,0) = q(0,0) = 0$ and $0 < \epsilon \ll 1$.
By Theorem 2.1, from $A(h) = 0$, we get:

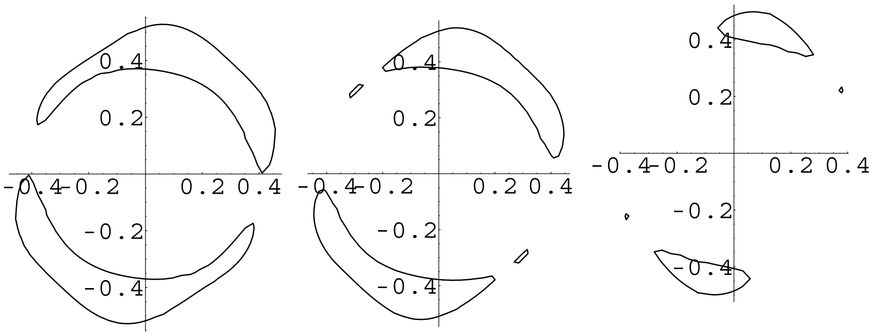

FIGURE 3. Orbits of type a) L_9 (left) b) L_{10}, L_{11} (middle) and c) L_{12}, L_{13} (right)

$$\lambda = \lambda(h) = \frac{\iint\limits_{C^h(D)} z(x,y)dxdy}{2\iint\limits_{C^h(D)} dxdy} \tag{8}$$

where $z(x,y) = xp'_x + yq'_y + p + q$.
This function $\lambda(h)$ is called *the detection function* of system (7).

From the Theorem 2.1 and using the detection function $\lambda(h)$ we obtain the following result [11]:

Proposition 3.1. *a) If $(h_0, \lambda(h_0))$ is an intersecting point of a line $\lambda = \lambda_0$ and the detection curve $\lambda = \lambda(h)$, with $\lambda'(h_0) > 0 (< 0)$, then the system (7) has only one stable (unstable) limit cycle nearby Γ^{h_0}; b) If the line $\lambda = \lambda_0$ and the detection curve $\lambda = \lambda(h)$ do not intersect each other, then the system (7) has no limit cycle. If the $\Gamma^h(D)$ shrinks as h increases, the stability of the limit cycle is opposite to the above cases.*

Let us consider now the equation $H(r,\theta) = h$ of the closed orbits studied above and denote:

$\alpha(\theta) = \frac{-18}{6-\cos 6\theta+\sin 6\theta-3\cos 2\theta+3\sin 2\theta}$,
$\beta(\theta) = \frac{6}{6-\cos 6\theta+\sin 6\theta-3\cos 2\theta+3\sin 2\theta}$,
$\gamma(\theta,h) = \frac{-6h}{6-\cos 6\theta+\sin 6\theta-3\cos 2\theta+3\sin 2\theta}$,
$g(\theta,h) = \sqrt{12\beta(\theta)^3 - 3\alpha(\theta)^2\beta(\theta)^2 + 81\gamma(\theta,h)^2 + s(\theta,h)}$, with
$s(\theta,h) = 12\alpha(\theta)^3\gamma(\theta,h) - 54\alpha(\theta)\beta(\theta)\gamma(\theta,h)$,
$f(\theta,h) = \sqrt[3]{36\alpha(\theta)\beta(\theta) - 108\gamma(\theta,h) - 8\alpha(\theta)^3 + 12g(\theta,h)}$.

Then the square of the roots of the equation $H(r,\theta) = h$ are:

$$r_1(\theta,h) = \tfrac{1}{6}f(\theta,h) - \tfrac{2}{3}\tfrac{3\beta(\theta)-\alpha(\theta)^2}{f(\theta,h)} - \tfrac{1}{3}\alpha(\theta),$$

$$r(\theta,h) = -\tfrac{1}{12}f(\theta,h) + \tfrac{1}{3}\tfrac{3\beta(\theta)-\alpha(\theta)^2}{f(\theta,h)} - \tfrac{1}{3}\alpha(\theta) + \tfrac{1}{2}i\sqrt{3}\left(\tfrac{1}{6}f(\theta,h) + \tfrac{2}{3}\tfrac{3\beta(\theta)-\alpha(\theta)^2}{f(\theta,h)}\right),$$

$$R(\theta,h) = -\tfrac{1}{12}f(\theta,h) + \tfrac{1}{3}\tfrac{3\beta(\theta)-\alpha(\theta)^2}{f(\theta,h)} - \tfrac{1}{3}\alpha(\theta) - \tfrac{1}{2}i\sqrt{3}\left(\tfrac{1}{6}f(\theta,h) + \tfrac{2}{3}\tfrac{3\beta(\theta)-\alpha(\theta)^2}{f(\theta,h)}\right).$$

Proposition 3.2. *In polar coordinates , system (7) becomes:*

$$\begin{aligned}\dot{r} &= -(g_1\cos\theta + g_2\sin\theta) - \varepsilon r\left[p(u_0)\cos^2\theta + q(u_0)\sin^2\theta - \lambda\right],\\ \dot{\theta} &= \tfrac{1}{r}\left(g_1\sin\theta + g_2\cos\theta\right) + \varepsilon\sin\theta\cos\theta(p(u_0)+q(u_0)),\end{aligned} \tag{9}$$

$u_0 = (r\cos\theta, r\sin\theta)$ *and the detection function (8) leads to:*

$$\lambda(h) = \frac{\int\limits_{\theta(h)}^{\bar{\theta}(h)}\int\limits_{r_2(\theta,h)}^{r_3(\theta,h)}(rp'_r + rq'_r - rp'_r\sin^2\theta - rq'_r\cos^2\theta + m + p + q)rdrd\theta}{2\int\limits_{\theta(h)}^{\bar{\theta}(h)}(r_3^2(\theta,h) - r_2^2(\theta,h))drd\theta}, \tag{10}$$

where $m = (q'_\theta - p'_\theta)\sin\theta\cos\theta$, $g_1 = \frac{\partial H(r\cos\theta, r\sin\theta)}{\partial y}$, $g_2 = \frac{\partial H(r\cos\theta, r\sin\theta)}{\partial x}$, $\theta(h), \bar{\theta}(h)$ *are, respectively, the minimum and the maximum of* θ *on the closed orbit* L^h *and* $r_2(\theta,h), r_3(\theta,h)$ *are the expressions of* L^h *in polar coordinates.*

Proof. We obtain (9) from (3) by changing $x = r\cos\theta, y = r\sin\theta$ by straightforward computations. On the other hand, since $p'_x = p'_r\cos\theta - \frac{1}{r}p'_\theta\sin\theta, q'_y = q'_r\sin\theta + \frac{1}{r}q'_\theta\cos\theta$, one gets

$xp'_x + yq'_y + p + q = rp'_r + rq'_r - rp'_r\sin^2\theta - p'_\theta\sin\theta\cos\theta - rq'_r\cos^2\theta + q'_\theta\sin\theta\cos\theta + p + q$. Now (10) is immediately. □

Consider perturbations of the following form:

$$p(x,y) = ux^n + vy^n + x^2y^{n-2}, q(x,y) = ux^n + vy^n - x^2y^{n-2}.$$

Then, taking into account the above considerations and detecting which root (r_1, r or R) corresponds to each closed orbits $L_i^h, i = 1-13$, we have the following theorem which describes the thirteen detection functions:

Theorem 3.1. *The corresponding thirteen detection functions associated to the thirteen closed orbits are:*

$$\lambda_i(h,u,v) = \frac{uI_i(h) + vJ_i(h)}{K_i(h)}, i = 1-13 \tag{11}$$

with

$$I_1(h)=\int\limits_{\theta_1(h)}^{\bar\theta_1(h)}\Big(r_1^{(n+2)/2}(\theta,h)-R^{(n+2)/2}(\theta,h)\Big)\cos^n\theta drd\theta;$$

$$J_1(h)=\int\limits_{\theta_1(h)}^{\bar\theta_1(h)}\Big(r_1^{(n+2)/2}(\theta,h)-R^{(n+2)/2}(\theta,h)\Big)\sin^n\theta drd\theta;$$

$$K_1(h)=2\int\limits_{\theta_1(h)}^{\bar\theta_1(h)}\left(r_1(\theta,h)-R(\theta,h)\right)drd\theta.$$

For $i=2,3,4,5$ and $i=9,...,13$ the corresponding integrals have the same forms as for $i=1$ but replacing (r_1,R) respectively by (R,r) $(i=2)$, (r_1,R) $(i=3)$, (R,r) $(i=4)$, (r_1,R) $(i=5)$ and (R,r), $(i=9,...,13)$. Of course, each integral is considered on the corresponding intervals $\theta_i(h),\bar\theta_i(h)$.

The other integrals are:

$$I_6(h)=\int\limits_0^{2\pi}R^{(n+2)/2}(\theta,h)\cos^n\theta drd\theta, J_6(h)=\int\limits_0^{2\pi}R^{(n+2)/2}(\theta,h)\sin^n\theta drd\theta$$

and $K_6(h)=2\int\limits_0^{2\pi}R(\theta,h)drd\theta.$

The integrals for $i=7,8$ have the same expressions as for $i=6$ but replacing, respectively, R by r_1 and by r, where $\theta_i(h),\bar\theta_i(h)$ are the minimum respectively the maximum of θ on L_i^h.

Proof. From (8) or (10) we have

$$\lambda(h)=\frac{\int\limits_{\theta(h)}^{\bar\theta(h)}\int\limits_{r_2(\theta,h)}^{r_3(\theta,h)}r^{n+1}(n+2)(u\cos^n\theta+v\sin^n\theta)drd\theta}{2\int\limits_{\theta(h)}^{\bar\theta(h)}(r_3^2(\theta,h)-r_2^2(\theta,h))drd\theta} \tag{12}$$

and from the forms of r_2,r_3 on each closed curve L_i^h we obtain the above integrals. □

4. Numerical illustrations of the detection functions and the distribution of limit cycles

In this paragraph we numerically compute the detection curves and the distribution of limit cycles. In doing that, for any fixed h we have to compute $\theta_i(h),\bar\theta_i(h)$. Because $\theta_i(h)$ has no explicit form we computed $\theta_i(h),\bar\theta_i(h)$ one by one from $\dot\theta=0$ and $H(r,\theta)=h$. We do not list all these values here because they would fill several pages. Once $\theta_i(h),\bar\theta_i(h)$ are found, the integrals can be computed numerically, and for a given h, they depend on u and v. On the other hand, for two given values of u and v, the detection curves can be plotted on the (h,λ)-plane, as illustrated in Fig.4. By Proposition 3.1 and the detection function graphs, the existence, number and distribution of limit cycles can then be obtained. We consider here only the case $n=4$, that corresponds to perturbations of quintic order.

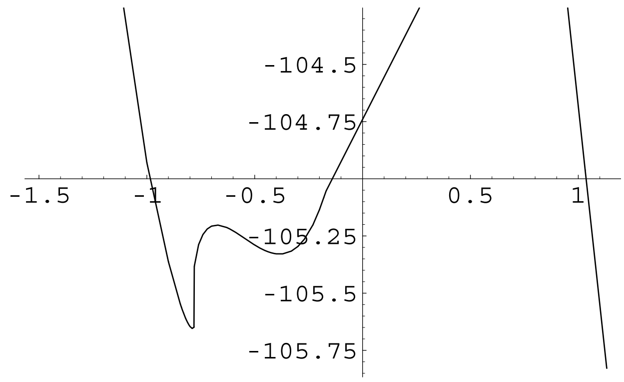

FIGURE 4. The detection function $\lambda_3(h), \lambda_5(h)$ and $\lambda_7(h)$ for $u = 0.19, v = -56.6$

Of a large number of cases which we considered, we stopped at the representative values $u = 0.19, v = -56.6$. From Proposition (3.1) and Fig.4 we get the following results:

Theorem 4.1. *Let* $u = 0.19, v = -56.6,\ 0 < \varepsilon \ll 1$. *Then*
a) If $-164.449 < \lambda < -140.958$*, system (3) has at least two limit cycles given by* L_2^h.
b) If $-140.958 < \lambda < -113.962$*, system (3) has at least three limit cycles given by* L_2^h *and* L_7^h.
c) If $-113.962 < \lambda < -105.654$*, system (3) has at least one limit cycle given by* L_7^h.
d) If $-105.654 < \lambda < -105.648$*, system (3) has at least five limit cycles given by* L_3^h *and* L_7^h.
e) If $-105.648 < \lambda < -105.383$*, system (3) has at least three limit cycles given by* L_3^h *and* L_7^h.
f) If $-105.383 < \lambda < -105.3281$*, system (3) has at least five limit cycles given by* L_3^h*,* L_5^h *and* L_7^h.
g) If $-105.3281 < \lambda < -105.207$*, system (3) has at least nine limit cycles, two of which given by* L_3^h*, six by* L_5^h *and one by* L_7^h.
h) If $-105.207 < \lambda < -105.053$*, system (3) has at least five limit cycles again given by* L_3^h*,* L_5^h *and* L_7^h.
k) If $-105.053 < \lambda < -93.4219$*, system (3) has at least three limit cycles given by* L_3^h *and* L_7^h.
l) If $-93.4219 < \lambda < -34.4849$*, system (3) has at least two limit cycles given by* L_3^h.

5. Conclusions

In this paper we have employed both qualitative and numerical procedures in order to study the existence, number and distribution of limit cycles of a Hamiltonian system under five-order perturbed terms. By numerical explorations we have drawn the shape of the graphs of the detection functions from which we described the distribution of the limit cycles. We also studied the system under perturbations of seven order ($n = 6$) but we have not observed that the distribution of the limit cycles is essentially different. Nevertheless, what can be said about the system under higher perturbations is an open question. However, an immediate intuitive conjecture given by this example is that the number of limit cycles of an n-degree polynomial differential system is of order kn for some natural numbers k.

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