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TRIGONOMETRIC SOLUTIONS FOR SOME ORDINERY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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Abstract

In this paper we construct approximate trigonometric solutions for some ordinary differential equations of second order with boundary conditions.

We look for trigonometric solutions for differential equations of the form

$$y'' + \alpha y' + \beta y + yy^n = F_0 + F_1 \cos x + F_2 \sin x \tag{1}$$

for $n = 2$ and $n = 3$, with boundary conditions

$$\left.\begin{aligned} y(2\pi) - y(0) = 0 \\ y'(2\pi) - y'(0) = 0 \end{aligned}\right\} \tag{2}$$

Remark 1. (1) *is a differential equation for some nonlinear oscillations.*

Here we find a solution of (1) in the form

$$y = A \sum_{n=0}^{\infty} \frac{\cos nx}{a^n} + B \sum_{n=1}^{\infty} \frac{\sin nx}{a^n} \tag{3}$$

when $x \in [0, 2\pi]$.

The following theorem holds:

Theorem 1. *The series* (3) *is a solution for the equation*

$$y'' + \alpha y' + \beta y + \gamma y^2 = F_1 \cos x$$

with boundary conditions (2) *if the next relations are satisfied:*

$$\begin{aligned}
&F_1 = 0\\
&B\beta + AB\gamma = 0\\
&2B\beta + A^2\gamma - B^2\gamma = 0\\
&B(a^2+1+\beta(1-a^2)) - \alpha A(1-a^2) = 0\\
&2A(1-a^2+\beta(3a^2+1)) - 2\alpha B(a^2+1) + 4aA^2\gamma = 0 \qquad (4)\\
&2B(4a^2+\beta(a^4+3a^2+1)) = 0\\
&A(1-a^4-\beta(5a^4+10a^2+1)-2a^4\gamma)+B\alpha(a^4+6a^2+1)-2A^2\gamma a^2(a^2+2)=0\\
&A(3a(a^2-1)+\beta(a^4+7a^2+4)+\gamma a^2(a^2+1))-3B\alpha(a^2+1)+A^2\gamma(3a^2+1)=0.
\end{aligned}$$

Proof. If $|a| > 1$, the series (3) is uniform convergent. With the transformations

$$\cos nx = \frac{e^{inx}+e^{-inx}}{2}, \quad \sin nx = \frac{e^{inx}-e^{-inx}}{2i},$$

we obtain that the sum of the series (3) is the function

$$y = \frac{a}{a^2+1-2a\cos x}(aA - A\cos x + B\sin x)\,. \qquad (5)$$

From (5) we have respectively:

$$y^2 = \frac{a^2}{(a^2+1-2a\cos x)^2}\Big(a^2A + \frac{A^2+B^2}{2} - 2aA^2\cos x + 2aAB\sin x + \frac{A^2-B^2}{2}\cos 2x - AB\sin 2x\Big), \qquad (6)$$

$$y' = \frac{a}{(a^2+1-2a\cos x)^2}[-2aB + B(a^2+1)\cos x + A(1-a^2)\sin x], \qquad (7)$$

$$y'' = \frac{a}{(a^2+1-2a\cos x)^3}[-3aA(1-a^2) + A(1-a^4)\cos x - B(a^4-6a^2+1)\sin x + aA(1-a^2)\cos 2x - aB(1+a^2)\sin 2x]\,. \qquad (8)$$

With the conditions (2), involving (5), (6), (7), (8) in (1) and comparing the coefficients next to the sin and cos we obtain the relations (4).

Corollary. *For $n = 2$, and $F_0 = F_1 = F_2 = 0$, the equation* (1) *has a solution in the form* (3) *if for a it is satisfied*

$$\begin{vmatrix} -a^8+9a^6+24a^4+7a^2+1 & a^8+9a^6+20a^4+9a^2+1 & -2 \\ 3a^7-4a^6+8a^5-16a^4-12a^2-a & -a^8+9a^6+24a^4+7a^2+1 & a^2+1 \\ -a^6-14a^4+8a^3-2a^2+1 & -a^6-4a^4-4a^2+1 & 0 \end{vmatrix} = 0$$

Remark 2. *With a substitution*

$$\omega t + \omega_0 = x$$

we transform the equation

$$y'' + \varphi y' + \psi y + \lambda y^2 = G_0 + G_1 \cos(\omega t + \omega_0) + G_2 \sin(\omega t + \omega_0)$$

in the form (1).

Remark 3. *Constructing the solution for the equation* (1) *in the form* (3), *we obtain it in a form of fraction rational function.*

We will illustrate the above procedure in some examples.

Example 1. The equation

$$y'' + \alpha y' + \beta y + \gamma y^3 = F_0 + F_1 \cos x + F_2 \sin x,$$

with boundary conditions (2), has the solution

$$y = A + A\sum_{n=1}^{\infty} \frac{\cos nx}{2^n} + B\sum_{n=1}^{\infty} \frac{\sin nx}{2^n} = \frac{2}{5 - 4\cos x}(2A - A\cos x + B\sin x)$$

if the next relations are satisfied:

$$\begin{aligned}
&F_1 = 0\\
&F_2 = 0\\
&4\beta B + 3\gamma BA^2 - \gamma\beta^3 = 0\\
&4\beta A + \gamma A^3 - 3\gamma AB^2 = 8F_0\\
&5(1+2\beta)B - 3\alpha A + 12\gamma A^2 B = 0\\
&3(-1+6\beta)A - 5\alpha B + 6\gamma A^3 - 6\gamma AB^2 = 30F_0\\
&-30\alpha A + (14+58\beta)B + 105\gamma A^2 B + 6\gamma B^3 = 0\\
&(15+117\beta)A - 41\alpha B + 51\gamma A^3 + 3\gamma AB^2 = 158F_0\\
&(36+172\beta)A - 60\alpha B + 88\gamma A^3 + 24\gamma AB^2 = 113F_0\,.
\end{aligned}$$

If $\alpha \neq 0$, and $\beta \neq \frac{8604}{7938} = 0,11289$, the equation (1) doesn't have a solution in the form (3).

Example 2. The equation

$$y'' \pm 3,4425339y' + 0,121289y - 2,3498979 \cdot 10^{-4} \cdot \frac{1}{F_0^2} y^3 = F_0$$

with boundary conditions (2), has the solution

$$y = 15,599792F_0 \sum_{n=0}^{\infty} \frac{\cos nx}{2^n} \pm 1,0136066F_0 \sum_{n=1}^{\infty} \frac{\sin nx}{2^n}.$$

Example 3. The equation

$$y'' - 0,5y + \gamma y^3 = 0$$

with boundary conditions (2), doesn't have a solution in the form (3).

Theorem 2. *The series*

$$y = A \sum_{n=0}^{\infty} \frac{\cos nx}{a^n} + B \sum_{n=1}^{\infty} \frac{\sin nx}{b^n} \tag{9}$$

is a solution for the equation (1) *with the boundary conditions* (2), *if it is satisfied the relations*

$$\varphi_i(\alpha, \beta, \gamma, a, b, A, B, F_0, F_1, F_2) = 0, \quad i = 1, 2, \ldots, 15 \quad \text{for} \quad n = 2 \tag{10}$$

$$\psi_i(\alpha, \beta, \gamma, a, b, A, B, F_0, F_1, F_2) = 0, \quad i = 1, 2, \ldots, 15 \quad \text{for} \quad n = 3. \tag{11}$$

Proof. If $|a| > 1$, $|b| > 1$ and $x \in [0, 2\pi]$, using the sums

$$1 + \frac{\cos x}{a} + \cdots + \frac{\cos nx}{a} + \cdots = \frac{a(a - \cos x)}{a^2 - 2a\cos x + 1}$$

$$\frac{\sin x}{b} + \frac{\sin 2x}{b^2} + \cdots + \frac{\sin nx}{b^n} + \cdots = \frac{b \sin x}{b^2 - 2b\cos x + 1}$$

we obtain that the series (9), is an uniform convergent towards the function:

$$y = \frac{Aa(ab^2 + a + b) - Aa(2ab + b^2 + 1)\cos x + Ba^2 b \sin x + Aab\cos 2x - Bab \sin 2x}{(a^2b^2 + (a+b)^2 + 1) - 2(a+b)(ab+1)\cos x + 2ab\cos 2x}, \tag{12}$$

i.e.

$$y = \frac{A_0 + A_1 \cos x + B_1 \sin x + A_2 \cos 2x + B_2 \sin 2x}{a_0 + a_1 \cos x + a_2 \cos 2x} = \\ = \frac{\sum_{i=0}^{2} A_i \cos ix + \sum_{i=1}^{2} B_i \sin ix}{\sum_{i=0}^{2} a_i \cos ix}.$$

From (9) we have respectively:

$$y' = \\ = \frac{C_0 + C_1 \cos x + D_1 \sin x + C_2 \cos 2x + D_2 \sin 2x + C_3 \cos 3x + D_3 \sin 3x + C_4 \cos 4x + D_4 \sin 4x}{(a_0 + a_1 \cos x + a_2 \sin 2x)^2}$$

$$= \frac{\sum_{i=0}^{4} C_i \cos ix + \sum_{i=1}^{4} D_i \sin ix}{\sum_{i=0}^{4} b_i \cos ix} \tag{13}$$

$$y'' = \frac{\sum_{i=0}^{6} E_i \cos ix + \sum_{i=1}^{6} F_i \sin ix}{(a_0 + a_1 \cos x + a_2 \cos 2x)^3} = \frac{\sum_{i=0}^{6} E_i \cos ix + \sum_{i=1}^{6} F_i \sin ix}{\sum_{i=0}^{6} c_i \cos ix} \tag{14}$$

$$y^2 = \frac{\sum_{i=0}^{4} G_i \cos ix + \sum_{i=1}^{4} H_i \sin ix}{\sum_{i=0}^{4} b_i \cos ix} \tag{15}$$

$$y^3 = \frac{\sum_{i=0}^{4} K_i \cos ix + \sum_{i=1}^{6} L_i \sin ix}{\sum_{i=0}^{6} c_i \cos ix} \tag{16}$$

where $A_i, B_i, C_i, D_i, E_i, F_i, G_i, H_i, K_i, L_i, a_i, b_i, c_i$ depend on a, b, A, B.

When we substitute (12), (13), (14), (15) and (16) in (1) and compare the coefficients next to the cos and sin we obtain the relations (10) for $n = 2$, and (11) for $n = 3$.

Remark 4. *If in the right side of* (1) *it is* $F_1 = 0$ *or* $F_2 = 0$ *then in* (10) *and* (11) *there are* 14 *relations, but, if* $F_1 = 0$ *and* $F_2 = 0$, *then in* (10) *and* (11) *there are* 13 *relations.*

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ТРИГОНОМЕТРИСКИ РЕШЕНИЈА ЗА НЕКОИ ОБИЧНИ ДИФЕРЕНЦИЈАЛНИ РАВЕНКИ ОД ВТОР РЕД

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Резиме

Во трудот се конструирани тригонометриски решенија на некои обични линеарни диференцијални равенки од втор ред при дадени гранични услови, и постапката е илустрирана на неколку примери.

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