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SERIES OF SINGLE SERVER SYSTEMS WITH BATCH ARRIVALS

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Abstract. In this paper we discuss a single server system with batch arrival with Poisson distribution for the interarrival time of the batches and exponential servicing time. The customers which arrive during the servicing time of one customer in the system, form another batch which leaves the system. In that way a new flow of customers is generated which is directed to another single server system. Continuing this procedure, we can define a series of single server systems. The main result of the paper is formulated in the theorem below, where we determine the distribution of the probabilities for the random variable X_{n+1} - the number of unserviced customers which leave the n^{th} system, as well as the first two moments and the variance.

It is studied in [1] the random variable Y - the size of the batch generated during the servicing of one customer in the systems M/G/1 and $M^X/G/1$. It is shown there that the distribution of Y in M/M/1 is geometric with parameter $p=1/(1+\rho)$, where $\rho=\lambda/\mu$ is the coefficient of the busy time of the system, while in the case of batch arrival with geometric distribution $P\{X=k\}=pq^{k-1}$, $k=1,2,\ldots$ for the size X of the batches, it is obtained that

$$P\{X=0\} = 1/(1+\rho), \quad \rho=\lambda/\mu,$$

$$P\{Y=k\} = \frac{\rho}{1+\rho}\cdot\frac{p}{1+\rho}[1-\frac{p}{1+\rho}]^{k-1}, \quad k=1,2,\ldots$$

When X has distribution $P\{X=k\} = pq^k$, $k=0,1,\ldots$, it is shown in [2] that

$$P\{Y=0\} = \frac{\mu}{\mu+\lambda q} = \frac{1}{1+\rho_1}, \quad \rho_1 = \frac{\lambda q}{\mu}$$

$$P\{Y=k\} = \frac{\lambda q}{\mu+\lambda q}\cdot\frac{\mu p}{\mu+\lambda q}[1-\frac{\mu p}{\mu+\lambda q}]^{k-1} =$$

$$= \frac{\rho_1}{1+\rho_1}\cdot\frac{p}{1+\rho_1}[1-\frac{p}{1+\rho_1}]^{k-1}, \quad k=1,2,\ldots$$

In this paper, using the results from [1] and [2], we shall propose a recurrent procedure in order to define a series of single server systems with batch arrivals and exponentially distributed servicing time. We find here the distribution of the size X_{n+1} for the batches of customers which leave the n^{th} system, as well as the moments EX_{n+1}, EX^2_{n+1} and the variance DX_{n+1}. In both cases we show that the size of the batches has a modified geometric distribution. This gives us an idea to introduce a recurrent procedure to generate flows with batch arrivals in the following way:

In the system $M_0/M/1$ with parameter λ_0 for the input process and parameter μ for the servicing time, there are generated batches of customers which arrive during the servicing time of one customer. Let us denote by X_1 the size of the first batch.

We define, now, a new system $M^X_1/M/1$ with one canal where batches of size X_1 arrive with a Poisson distribution with parameter λ_1 for the arriving moments of the batches and an exponential servicing with parameter μ. By X_2 we denote the random variable which represents the number of the customers arriving in this system during the servicing time of one customer.

In the third system $M^X_2/M/1$, the quasi-Poisson process has parameter λ_2 and size X_2 of the batches, and the servicing is the same as before. This system leave batches of size X_3, the distribution of which is used to define the fourth system and the random variable X_4, and so on.

Generally, if X is the random variable - the number of customer which arrive during the servicing time of one customer in the system $M^X_{n-1}/M/1$ with parameters λ_{n-1} and μ for the input process and servicing time, respectively, then the following system $M^X_n/M/1$ has a quasi-Poisson flow with parameter λ_n and the size X_n of the batches, n=1,2,... . In that way we generate a series of quasi-Poisson flows where the distribution of X_n is a modified geometric one. Let us prove, now, the following

<u>Theorem</u>. If λ_n, n=0,1,2,... , is the parameter of the input quasi-Poisson process in the n^{th} system and μ the parameter of the exponential servicing, then:

a) for the distribution of the probabilities of X_{n+1} - the number of the unserviced customers in the n^{th} system we have that

$$P\{X_{n+1}=0\} = \frac{1+\rho_o+\rho_o\rho_1+\ldots+\rho_o\rho_1\ldots\rho_{n-1}}{1+\rho_o+\rho_o\rho_1+\ldots+\rho_o\rho_1\ldots\rho_n}, \quad \rho_1 = \frac{\lambda_1}{\mu}, \quad i=0,1,\ldots$$

$$P\{X_{n+1}=k\} = \frac{\rho_o\rho_1\ldots\rho_n}{(1+\rho_o+\rho_o\rho_1+\ldots+\rho_o\rho_1\ldots\rho_n)^2}\cdot$$

$$\cdot[1-\frac{1}{1+\rho_o+\rho_o\rho_1+\ldots+\rho_o\rho_1\ldots\rho_n}]^{k-1}, \quad k=1,2,\ldots$$

b) the first two moments and the variance of X_{n+1} are

$$EX_{n+1} = \prod_{i=o}^{n} \rho_i,$$

$$EX^2_{n+1} = \prod_{i=o}^{n} \rho_i(1+2\rho_o+\ldots+2\rho_o\rho_1\ldots\rho_n),$$

$$DX_{n+1} = \prod_{i=o}^{n} \rho_i(1+2\rho_o+\ldots+2\rho_o\rho_1\ldots\rho_{n-1}+\rho_o\rho_1\ldots\rho_n)$$

Proof. For n=1, it is shown in [1] that the distribution of X_1 and the moments and variance are

$$P\{X_1=k\} = \frac{1}{1+\rho_o}[1+\frac{1}{1+\rho_o}]^k, \quad \rho_o = \frac{\lambda_o}{\mu}, \quad k=0,1,2,\ldots$$

$$EX_1=\rho_o, \quad EX_1^2=\rho_o(2\rho_o+1), \quad DX_1=\rho_o(\rho_o+1)$$

a) Let n=2; the generating function of X_2 is given by

$$Q_2(z) = \beta(\lambda_1-\lambda_1 Q_1(z))$$

where $Q_1(z)$ is the generating function of X_1. After some transformations over $Q_2(z)$ in order to bring its expression to an appropriated form, we can get

$$P\{X_1=0\} = \frac{1+\rho_o}{1+\rho_o+\rho_o\rho_1}, \quad \rho_1 = \frac{\lambda_1}{\mu}$$

$$P\{X_1=k\} = \frac{\rho_o\rho_1}{(1+\rho_o+\rho_o\rho_1)^2}[1-\frac{1}{1+\rho_o+\rho_o\rho_1}]^{k-1}, \quad k=1,2,\ldots$$

$$EX_2 = \rho_o\rho_1$$

$$EX_2^2 = \rho_o\rho_1(2\rho_o\rho_1+2\rho_o+1)$$

$$DX_2 = \rho_o\rho_1(\rho_o\rho_1+2\rho_o+1)$$

If we denote by $Q_n(z)$ the generating function of the random variable X_n, then

$$Q_o(z) = z$$

$$Q_{n+1}(z) = \beta(\lambda_n - \lambda_n Q_n(z)), \quad n=0,1,2,\ldots \qquad (*)$$

where $\beta(s)$ is the Laplace-Stiltjes transform of the exponential distribution with parameter μ.

Let us put

$$c_o = 1, \quad c_{n+1} = c_n + \prod_{i=o}^{n} \rho_i$$

Then for the distribution of the probabilities of X_{n+1} we have that

$$P\{X_{n+1}=0\} = \frac{c_n}{c_{n+1}}$$

$$P\{X_{n+1}=k\} = \frac{c_{n+1}-c_n}{c_{n+1}^2}[1-\frac{1}{c_{n+1}}]^{k-1}, \quad k=1,2,\ldots$$

and for the generating function of X_{n+1},

$$Q_{n+1}(z) = \frac{c_n+(c_n-1)z}{c_{n+1}+(c_{n+1}-1)z}$$

Further we proceed by induction.

b) If we differentiate (*) necessary number of times and if we make use of the relations which exist between the differentiates of the Laplace-Stiltjes transform, i.e. of the generating functions of the distributions and the corresponding moments, we can get the recurrent relations as follows

$$EX_{n+1} = \rho_n EX_n$$

$$EX_{n+1}^2 = \rho_n EX_n^2 + 2\rho_n^2 (EX_n)^2$$

$$DX_{n+1} = \rho_n EX_n^2 + \rho_n^2 (EX_n)^2$$

from where follow the expressions for the moments.

Observe that the distribution of the output process is of the same type as the one of the input process which has generating function for the size of the batches equal to $\Phi_1 = pz/(1-qz)$, and the parameter for the arrival time $\lambda_1 = \lambda q$.

R E F E R E N C E S

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СЕРИЈА ЕДНОКАНАЛНИ СИСТЕМИ СО ГРУПНО ПРИСТИГНУВАЊЕ

М. Георгиева

Р е з и м е

Во овој труд се дефинира, на рекурентен начин, серија од едноканални системи со групно пристигнување на клиентите во моменти што образуваат Пуасонов поток и со експоненцијално распределено време на обслужување. При предпоставка дека распределбата на бројот на клиенти во група за влезниот поток во првиот систем е геометриска, се наоѓа распределбата на веројатностите на случајната променлива X_{n+1} - број на клиенти што го напуштаат n-тиот систем (n=1,2,...), првите два момента на оваа случајна променлива како и нејзината дисперзија.

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