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ON SOME ČEBYŠEV–GRÜSS TYPE INTEGRAL INEQUALITIES

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Abstract

In this paper we establish some Čebyšev–Grüs type integral inequalities, by using the extension of the Montgomery identity.

1. Introduction

For two absolutely continuous functions $f, g\colon [a,b] \to R$ consider the functional

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right),$$

where the involved integrals exist.

In 1882, Čebyšev [4] proved that if $f', g', \in L_\infty[a,b]$, then

$$|T(f,g)| \le \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty\,. \tag{1.1}$$

In 1935, Grüss [5] showed that

$$|T(f,g)| \le \frac{1}{4}(M-m)(N-n)\,, \tag{1.2}$$

for every two integrable functions $f, g\colon [a,b] \to R$ satisfying the condition

$$m \le f(x) \le M, \quad n \le g(x) \le N, \quad \forall\, x \in [a,b]$$

where m, M, n, N are given real constants.

Let $f\colon [a,b] \to R$ be differentiable on $[a,b]$ and $f'\colon [a,b] \to R$ is integrable on $[a,b]$. Then the Montgomery identity holds [10]

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\int_a^b P(x,t)f'(t)\,dt\,, \tag{1.3}$$

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t)=\begin{cases}\dfrac{t-a}{b-a}\,, & a\le t\le x\\ \dfrac{t-b}{b-a}\,, & x<t\le b\,.\end{cases} \tag{1.4}$$

Let $w\colon[a,b]\to[0,\infty)$ be some probability density function, that is an integrable function satisfying $\int_a^b w(t)\,dt=1$, and $W(t)=\int_a^t w(x)\,dx$ for $t\in[a,b]$, $W(t)=0$ for $t<a$, and $W(t)=1$ for $t>b$. In [10] Pečarić has given the following weighted of the Montgomery identity:

$$f(x)=\int_a^b w(t)f(t)dt+\int_a^b P_w(x,t)f'(t)dt, \tag{1.5}$$

where $P_w(x,t)$ is the weighted Peano kernel defined by

$$P_w(x,t)=\begin{cases}W(t), & a\le t\le x\\ W(t)-1, & x<t\le b\,.\end{cases} \tag{1.6}$$

We use the following notation to simplify the details of the presentation. For some suitable functions $w,f,g\colon[a,b]\to R$, we set

$$T(w,f,g)=\int_a^b w(x)f(x)g(x)dx-\left(\int_a^b w(x)f(x)dx\right)\left(\int_a^b w(x)g(x)dx\right), \tag{1.7}$$

and define $\|\cdot\|_\infty$ as the usual Lebesgue norm on $L_\infty[a,b]$ that is, $\|h\|_\infty:=ess\sup_{t\in[a,b]}|h(t)|$ for $h\in L_\infty[a,b]$.

In the paper [9] B.G. Pachpatte by using Pečarić's extension of the Montgomery identity (1.5) proved the following two theorems.

Theorem A. Let $f,g\colon[a,b]\to R$ be differentiable on $[a,b]$ and $f',g'\colon[a,b]\to R$ are integrable on $[a,b]$. Let $w\colon[a,b]\to[0,+\infty)$ be an integrable function satisfying $\int_a^b w(t)dt=1$.

Then

$$|T(w,f,g)|\le\|f'\|_\infty\|g'\|_\infty\int_a^b w(x)H^2(x)dx \tag{1.8}$$

where

$$H(x)=\int_a^b|P_w(x,t)|dt \tag{1.9}$$

for $x\in[a,b]$ and $P_w(x,t)$ is the weighted Peano kernel given by (1.6).

Theorem B. Let $f, g\colon [a,b] \to R$ be differentiable on $[a,b]$ and $f', g'\colon [a,b] \to R$ are integrable on $[a,b]$. Let $w\colon [a,b] \to [0,+\infty)$ be an integrable function satisfying $\int_a^b w(t)dt = 1$. Then

$$|T(w,f,g)| \leq \frac{1}{2}\int_a^b w(x)\big[|g(x)|\,\|f'\|_\infty + |f(x)|\,\|g'\|_\infty\big]H(x)dx \tag{1.10}$$

where $H(x)$ is defined by (1.9).

The main purpose of this paper is to generalize of the Theorem A. and Theorem B. In fact, by using the extension Montgomery identity obtained in [2] and [3], we get inequalities similar to those of Čebyšev and Grüss inequalities. Our results in special case yields to the results B. G Pachpatte (1.8) and (1.10).

1. Main results

In this section our main results are given. In what follows, we denote by R, the set of real numbers and $[a,b] \subset R$, $a < b$. We use the usual convetion that on empty sum is taken to be zero.

Theorem 1. Let (p,q) be a pair of conjugate exponents, i.e. $1 \leq p,\ q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g\colon [a,b] \to R$ functions such that $f^{(n)}, g^{(n)} \in L_p[a,b]$. Let $w\colon [a,b] \to [0,+\infty)$ be an integrable function satisfying $\int_a^b w(t)dt = 1$. Then for $n \geq 2$ the following inequality holds

$$\begin{aligned}
&\Big|T(w,f,g) + \int_a^b w(x)\Big[f(x) - \int_a^b w(t)f(t)dt\Big]\times \\
&\times \Big[\sum_{i=0}^{n-2} \frac{g^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big]dx \\
&+ \int_a^b w(x)\Big[g(x) - \int_a^b w(t)g(t)dt\Big]\Big[\sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big]dx \\
&+ \int_a^b w(x)\Big[\sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big]\times \\
&\times \Big[\sum_{i=0}^{n-2} \frac{g^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big]dx\Big| \\
&\leq \frac{\|f^{(n)}\|_p \|g^{(n)}\|_p}{\big[(n-1)!\big]^2}\int_a^b w(x)\big[\|T_{w,n}(x,s)\|_q\big]^2 dx
\end{aligned} \tag{2.1}$$

where

$$T_{w,n}(x,s) = \begin{cases} \int_a^s w(u)(u-s)^{n-1}du\,, & a \le s \le x \\ -\int_s^b w(u)(u-s)^{n-1}du, & x < s \le b\,. \end{cases} \tag{2.2}$$

Proof. The following integral identity is valid (see [2]):

$$\begin{aligned} f(x) = &\int_a^b w(t)f(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^{i+1}ds \\ &+ \frac{1}{(n-1)!} \int_a^b T_{w,n}(x,s)f^{(n)}(s)ds, \end{aligned} \tag{2.3}$$

and similarly

$$\begin{aligned} g(x) = &\int_a^b w(t)g(t)dt - \sum_{i=0}^{n-2} \frac{g^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^{i+1}ds \\ &+ \frac{1}{(n-1)!} \int_a^b T_{w,n}(x,s)g^{(n)}(s)ds, \end{aligned} \tag{2.4}$$

From (2.3) and (2.4) we have

$$\begin{aligned} &\left[f(x) - \int_a^b w(t)f(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^{i+1}ds\right] \times \\ &\times \left[g(x) - \int_a^b w(t)g(t)dt + \sum_{i=0}^{n-2} \frac{g^{(i+1)}(x)}{(1+i)!} \int_a^b w(s)(x-a)^{i+1}ds\right] \\ &= \frac{1}{\left[(n-1)!\right]^2} \left[\int_a^b T_{w,n}(x,s)f^{(n)}(s)ds\right] \left[\int_a^b T_{w,n}(x,s)g^{(n)}(s)ds\right], \end{aligned}$$

i.e.,

$$\begin{aligned} &f(x)g(x) - f(x)\int_a^b w(t)g(t)dt - g(x)\int_a^b w(t)f(t)dt + \left(\int_a^b w(t)f(t)dt\right)\left(\int_a^b w(t)g(t)dt\right) \\ &+ \left[f(x) - \int_a^b w(t)f(t)dt\right]\left[\sum_{i=0}^{n-2} \frac{g^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^{i+1}ds\right] \\ &+ \left[g(x) - \int_a^b w(t)g(t)dt\right]\left[\sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^{i+1}ds\right] \\ &+ \left[\sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^{i+1}ds\right]\left[\sum_{i=0}^{n-2} \frac{g^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^{i+1}ds\right] \\ &= \frac{1}{\left[(n-1)!\right]^2} \int_a^b T_{w,n}(x,s)f^{(n)}(s)ds\Big] \left[\int_a^b T_{w,n}(x,s)g^{(n)}(s)ds\right]. \end{aligned} \tag{2.5}$$

Multiplying both sides of (2.5) by $w(x)$ and then integrating both sides of the resulting identity with respect to x from a to b and using the fact that $\int_a^b w(t)dt = 1$, we have

$$\begin{aligned}
&T(w,f,g)+\int_a^b w(x)\Big[f(x)-\int_a^b w(t)f(t)dt\Big]\Big[\sum_{i=0}^{n-2}\frac{g^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big]dx\\
&+\int_a^b w(x)\Big[g(x)-\int_a^b w(t)g(t)dt\Big]\Big[\sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big]dx\\
&+\int_a^b w(x)\Big[\sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big] \qquad (2.6)\\
&\Big[\sum_{i=0}^{n-2}\frac{g^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big]dx\\
&=\frac{1}{\big[(n-1)!\big]^2}\int_a^b w(x)\Big[\int_a^b T_{w,n}(x,s)f^{(n)}(s)ds\Big]\Big[\int_a^b T_{w,n}(x,s)g^{(n)}(s)ds\Big]dx\,.
\end{aligned}$$

From (2.6) and using the properties of modulus we observe that

$$\begin{aligned}
&\Big|T(w,f,g)+\int_a^b w(x)\Big[f(x)-\int_a^b w(t)f(t)dt\Big]\times\\
&\times\Big[\sum_{i=0}^{n-2}\frac{g^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big]dx\\
&+\int_a^b w(x)\Big[g(x)-\int_a^b w(t)g(t)dt\Big]\Big[\sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big]dx\\
&+\int_a^b w(x)\Big[\sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big]\times \qquad (2.7)\\
&\times\Big[\sum_{i=0}^{n-2}\frac{g^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big]dx\Big|\\
&\le\frac{1}{\big[(n-1)!\big]^2}\int_a^b w(x)\Big[\int_a^b |T_{w,n}(x,s)f^{(n)}(s)ds\Big]\Big[\int_a^b |T_{w,n}(x,s)g^{(n)}(s)|ds\Big]dx\,.
\end{aligned}$$

After applying Hölder inequality

$$\int_a^b |T_{w,n}(x,s)f^{(n)}(s)|ds \le \|T_{w,n}(x,s)\|_q\ \|f^{(n)}\|_p$$

$$\int_a^b |T_{w,n}(x,s)g^{(n)}(s)|ds \le \|T_{w,n}(x,s)\|_q\ \|g^{(n)}\|_p$$

we obtain the inequality (2.1).

Remark 1. For n=1 identity (2.3) reduces to weighted Montgomery identity

$$f(x)=\int_a^b w(t)f(t)dt+\int_a^b T_w(x,s)f'(s)ds$$

where T_w is the weighted Peano kernel defined by (1.6).

So, if take $p=\infty$, $q=1$ and $n=1$ the inequality (2.1) reduces to

$$|T(w,f,g)|\le\int_a^b w(x)\big[\,\|T_w(x,s)\|_1\big]^2\,\big[\,\|f'\|_\infty\|g'\|_\infty\big]dx$$

where

$$\|T_w(x,s)\|_1=\int_a^b |T_w(x,s)|ds=H(x)\,.$$

Therefore the inequality from the last theorem reduces to inequality (1.8) obtained by B.G. Pachpatte, i.e.:

$$|T(w,f,g)|\le\|f'\|_\infty\|g'\|_\infty\int_a^b w(x)H^2(x)dx\,.$$

Corollary 1. Suppose all the assumptions from the Theorem 1. holds, and $w(t)=\frac{1}{b-a}$, $t\in[a,b]$. Then

$$\begin{aligned}
&\Big|T(f,g)+\frac{1}{b-a}\int_a^b\Big[f(x)-\frac{1}{b-a}\int_a^b f(t)dt\Big]\times\\
&\times\Big[\sum_{i=0}^{n-2}g^{(i+1)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!(b-a)}\Big]dx\\
&+\frac{1}{b-a}\int_a^b\Big[g(x)-\frac{1}{b-a}\int_a^b g(t)dt\Big]\Big[\sum_{i=0}^{n-2}f^{(i+1)}(x)\,\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!(b-a)}\Big]dx\\
&+\frac{1}{b-a}\int_a^b\Big[\sum_{i=0}^{n-2}f^{(i+1)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!(b-a)}\Big]\times \qquad (2.8)\\
&\times\Big[\sum_{i=0}^{n-2}g^{(i+1)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!(b-a)}\Big]dx\Big|\\
&\le\begin{cases}
\dfrac{1}{(n!)^2}\,\dfrac{\|f^{(n)}\|_p\|g^{(n)}\|_p}{(b-a)^3}\displaystyle\int_a^b\Big(\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\Big)^{2/q}dx, & 1<p<\infty\,,\\[2ex]
\dfrac{2(b-a)^{2n}\big[1+(2n+3)B(n+2,n+2)\big]}{(n!)^2(n+1)^2(2n+3)}\,\|f^{(n)}\|_\infty\|g^{(n)}\|_\infty\,, & p=\infty\,,\\[2ex]
\dfrac{(b-a)^{2n-2}(2^{2n+1}-1)}{(n!)^2 2^{2n}(2n+1)}\|f^{(n)}\|_1\|g^{(n)}\|_1\,, & p=1\,.
\end{cases}
\end{aligned}$$

Proof. (I) For $1 < p < \infty$ we have

$$\begin{aligned}
\|T_n(x,s)\|_q &= \Big(\int_a^b |T_n(x,s)|^q ds\Big)^{1/q} \\
&= \Big(\int_a^x \Big|\frac{-(a-s)^n}{n(b-a)}\Big|^q ds + \int_x^b \Big|\frac{-(b-s)^n}{n(b-a)}\Big|^q ds\Big)^{1/q} \\
&= \Big(\frac{(x-a)^{nq+1} + (b-x)^{nq+1}}{n^q(b-a)^q(nq+1)}\Big)^{1/q} \\
&= \frac{1}{n(b-a)}\Big(\frac{(x-a)^{nq+1} + (b-x)^{nq+1}}{nq+1}\Big)^{1/q}.
\end{aligned}$$

If we apply the inequality (2.1), the first inequality (2.8) follows.

(II) For $p = \infty$ we have

$$\begin{aligned}
\|T_n(x,s)\|_1 &= \int_a^b |T_n(x,s)| ds = \int_a^x \Big|\frac{-(a-s)^n}{n(b-a)}\Big| ds + \int_x^b \Big|\frac{-(b-s)^n}{n(b-a)}\Big| ds \\
&= \frac{1}{n(n+1)(b-a)}\big((x-a)^{n+1} + (b-x)^{n+1}\big).
\end{aligned}$$

Now, we have

$$\begin{aligned}
&\int_a^b \big(\|T_n(x,s)|_1\big)^2 dx = \frac{1}{n^2(n+1)^2(b-a)^2}\int_a^b \big((x-a)^{n+1} + (b-x)^{n+1}\big)^2 dx \\
&= \frac{1}{n^2(n+1)^2(b-a)^2}\Big[\int_a^b (x-a)^{2n+2}dx + 2\int_a^b (x-a)^{n+1}(b-x)^{n+1}dx \\
&+ \int_a^b (b-x)^{2n+2}dx\Big].
\end{aligned}$$

We have

$$\int_a^b (x-a)^{2n+2}dx = \frac{(b-a)^{2n+3}}{2n+3}, \quad \int_a^b (b-x)^{2n+2}dx = \frac{(b-a)^{2n+3}}{2n+3},$$

and, using substitution $x - a = u(b-a)$ the other integral is equal to

$$2\int_a^b (x-a)^{n+1}(b-x)^{n+1}dx = 2(b-a)^{2n+3}B(n+2,n+2).$$

So, we have

$$\int_a^b \Big(\|T_n(x,s)\|_1\Big)^2 dx = \frac{2(b-a)^{2n}}{(n!)^2(n+1)^2(2n+3)}\big[1 + (2n+3)B(n+2,n+2)\big],$$

and we obtain the second inequality (2.8).

(III) For $p=1$, we have

$$\|T_n(x,s)\|_\infty = \sup_{x\in[a,b]} |T_n(x,s)| = \max\left\{ \sup_{s\in[a,x]} \left|\frac{-a(a-s)^n}{n(b-a)}\right|, \sup_{s\in[x,b]} \left|\frac{-(b-s)^n}{n(b-a)}\right| \right\}.$$

By simple calculation we get

$$\sup_{s\in[a,x]} \left|\frac{-(a-s)^n}{n(b-a)}\right| = \frac{(x-a)^n}{n(b-a)}, \quad \sup_{s\in[x,b]} \left|\frac{-(b-s)^n}{n(b-a)}\right| = \frac{(b-x)^n}{n(b-a)}.$$

So, we get

$$\|T_n(x,s)\|_\infty = \sup_{x\in[a,b]} |T_n(x,s)| = \max\left\{ \frac{(b-x)^n}{n(b-a)}, \frac{(x-a)^n}{n(b-a)} \right\}.$$

Now, we have

$$\int_a^{(a+b)/2} \left[\frac{(b-x)^n}{n(b-a)}\right]^2 dx = \frac{(b-a)^{2n-1}(2^{2n+1}-1)}{2^{2n+1}(2n+1)n^2},$$

$$\int_{(a+b)/2}^{a} \left[\frac{(x-a)^n}{n(b-a)}\right]^2 dx = \frac{(b-a)^{2n-1}(2^{2n+1}-1)}{2^{2n+1}(2n+1)n^2}.$$

So, we obtain

$$\int_a^b \left(\|T_n(x,s)\|_\infty\right)^2 dx = \frac{(b-a)^{2n-1}(2^{2n+1}-1)}{2^{2n}(2n+1)n^2}.$$

Then, if we apply the inequality (2.1), we obtain the third inequality (2.8).

Remark 2. For $n=1$ the inequalities from the Corollary 1. reduces to

$$|T(f,g)| \le \begin{cases} \dfrac{\|f'\|_p\|g'\|_p}{(b-a)^3} \int_a^b \left(\dfrac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right)^{2/q} dx\,, & 1<p<\infty \\ \dfrac{7}{60}(b-a)^2\, \|f'\|_\infty \|g'\|_\infty\,, & p=\infty \\ \dfrac{7}{12}\|f'\|_1\|g'\|_1\,, & p=1\,. \end{cases}$$

For n=2, the inequalities from the Corollary 1. reduces to

$$\Big|T(f,g)+\frac{1}{b-a}\int_a^b\Big[f(x)-\frac{1}{b-a}\int_a^b f(t)dt\Big]\Big[g'(x)\frac{(b-x)^2-(a-x)^2}{2!(b-a)}\Big]dx$$
$$+\frac{1}{b-a}\int_a^b\Big[g(x)-\frac{1}{b-a}\int_a^b g(t)dt\Big]\Big[f'(x)\frac{(b-x)^2-(a-x)^2}{2!(b-a)}\Big]dx$$
$$+\frac{1}{b-a}\int_a^b\Big[f'(x)\frac{(b-x)^2-(a-x)^2}{2!(b-a)}\Big]\Big[g'(x)\frac{(b-x)^2-(a-x)^2}{2!(b-a)}\Big]dx\Big|$$
$$\leq\begin{cases}\frac{1}{4}\frac{1}{(b-a)^3}\|f''\|_p\|g''\|_p\int_a^b\Big(\frac{(x-a)^{2q+1}+(b-x)^{2q+1}}{2q+1}\Big)^{2/q}dx, & 1<p<\infty\\ \frac{(b-a)^4}{120}\|f''\|_\infty\|g''\|_\infty, & p=\infty\\ \frac{31(b-a)^2}{320}\|f''\|_1\|g''\|_1, & p=1.\end{cases}$$

Theorem 2. Let (p,q) be a pair of conjugate exponents i.e. $1\leq p,\ q\leq\infty$ and $\frac{1}{p}+\frac{1}{q}=1$, and $f,g\colon[a,b]\to R$ function such that $f^{(n)},g^{(n)}\in L_p[a,b]$. Let $w\colon[a,b]\to[0,+\infty)$ be an integrable function satisfying $\int_a^b w(t)dt=1$. Then for $n\geq 2$ the following inequality holds

$$\Big|T(w,f,g)+\frac{1}{2}\int_a^b w(x)\Big[g(x)\sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds$$
$$+f(x)\sum_{i=0}^{n-2}\frac{g^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big]dx\Big| \tag{2.9}$$
$$\leq\frac{1}{2(n-1)!}\int_a^b w(x)\|T_{w,n}(x,s)\|_q\big[\,|g(x)|\,\|f^{(n)}\|_p+|f(x)|\,\|g^{(n)}\|_p\big]dx$$

where $T_{w,n}(x,s)$ is given by (2.2).

Proof. Multiplying both sides of (2.3) and (2.4) by $w(x)g(x)$and $w(x)f(x)$, adding the resulting identities and rewriting we have

$$w(x)f(x)g(x)-\frac{1}{2}\Big[w(x)g(x)\int_a^b w(t)f(t)dt+w(x)f(x)\int_a^b w(t)g(t)dt\Big]$$
$$+\frac{1}{2}\Big[w(x)g(x)\sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds+$$

$$+ w(x)f(x)\sum_{i=0}^{n-2}\frac{g^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big] \tag{2.10}$$
$$= \frac{1}{2(n-1)!}\Big[w(x)g(x)\int_a^b T_{w,n}(x,s)f^{(n)}(s)ds + w(x)f(x)\int_a^b T_{w,n}(x,s)g^{(n)}(s)ds\Big].$$

Integrating both sides of (2.10) with respect to x from a to b and rewriting we have

$$\begin{aligned} &T(w,f,g) + \frac{1}{2}\int_a^b \Big[w(x)g(x)\sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds \\ &+ w(x)f(x)\sum_{i=0}^{n-2}\frac{g^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big]dx \\ &= \frac{1}{2(n-1)!}\int_a^b \Big[w(x)g(x)\int_a^b T_{w,n}(x,s)f^{(n)}(s)ds \\ &+ w(x)f(x)\int_a^b T_{w,n}(x,s)g^{(n)}(s)ds\Big]dx\,. \end{aligned} \tag{2.11}$$

From (2.11) and using the properties of modulus we observe that

$$\begin{aligned} &\Big|T(w,f,g) + \frac{1}{2}\int_a^b w(x)\Big[g(x)\sum_{i=0}^{n-2}\frac{f^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds \\ &+ f(x)\sum_{i=0}^{n-2}\frac{g^{(i+1)}(x)}{(i+1)!}\int_a^b w(s)(s-x)^{i+1}ds\Big]dx\Big| \\ &\leq \frac{1}{2(n-1)!}\int_a^b \Big[w(x)g(x)\int_a^b |T_{w,n}(x,s)f^{(n)}(s)|ds \\ &+ |f(x)|\int_a^b |T_{w,n}(x,s)g^{(n)}(s)|ds\Big]dx\,. \end{aligned} \tag{2.12}$$

After applying Hölder inequality

$$\int_a^b |T_{w,n}(x,s)f^{(n)}(s)|ds \leq \|T_{w,n}(x,s)\|_q \|f^{(n)}(s)\|_p$$
$$\int_a^b |T_{w,n}(x,s)g^{(n)}(s)|ds \leq \|T_{w,n}(x,s)\|_q \|g^{(n)}\|_p$$

we obtain the inequality (2.9).

Remark 3. For $n = 1$ identity (2.3) reduces to weighted Montgomery identity

$$f(x) = \int_a^b w(t)f(t)dt + \int_a^b T_w(x,s)f'(s)ds$$

where T_w is weighted Peano kernel defined by (1.6).

So, if we take $p = \infty$, $q = 1$ and $n = 1$ the inequality (2.9) reduces to

$$|T(w,f,g)| \leq \frac{1}{2}\int_a^b w(x)\big[|g(x)|\, \|T_w(x,s)\|_1\, \|f'\|_\infty + |f(x)|\, \|T_w(x,s)\|_1\, \|g'\|_\infty\big]dx$$

where

$$\|T_w(x,s)\|_1 = \int_a^b |T_w(x,s)|ds = H(x)\,.$$

Therefore, the inequality from the last theorem reduces to inequality (1.10) obtained by B.G. Pachpatte, i.e.:

$$|T(w,f,g)| \leq \frac{1}{2}\int_a^b w(x)\big[\,|g(x)|\, \|f'\|_\infty + |f(x)|\, \|g'\|_\infty\big]H(x)dx\,.$$

Corollary 2. Suppose all the assumptions from the Theorem 2. holds, and $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. Then

$$\begin{aligned}
&\Big|T(f,g) + \frac{1}{2(b-a)}\int_a^b \Big[g(x)\sum_{i=0}^{n-2} f^{(i+1)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!(b-a)} \\
&+ f(x)\sum_{i=0}^{n-2} g^{(i+1)}(x)\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!(b-a)}\Big]dx\Big| \qquad (2.13)\\
&\leq \begin{cases}
\frac{1}{2n!}\,\frac{1}{(b-a)^2}\Big(\frac{(b-a)^{nq+1}}{nq+1}\Big)^{1/q}\Big[\,\|f^{(n)}\|_p\|g\|_1 + \|g^{(n)}\|_p\|f\|_1\Big], & 1<p<\infty\\
\frac{(b-a)^{n-1}}{2n!(n+1)}\big[\,\|f^{(n)}\|_\infty\|g\|_1 + \|g^{(n)}\|_\infty\|f\|_1\big]\,, & p=\infty\\
\frac{(b-a)^{n-2}}{2n!}\|f^{(n)}\|_1\|g\|_1 + \|g^{(n)}\|_1\|f\|_1\,, & p=1
\end{cases}
\end{aligned}$$

Proof. (I) For $1 < p < \infty$, we have

$$\begin{aligned}
\|T_n(x,s)\|_q &= \frac{1}{n(b-a)}\Big(\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\Big)^{1/q}\\
&\leq \frac{1}{n(b-a)}\Big(\frac{(b-a)^{nq+1}}{nq+1}\Big)^{1/q}.
\end{aligned}$$

Now, we have

$$\int_a^b \left[|g(x)| \, \|f^{(n)}\|_p + |f(x)| \, \|g^{(n)}\|_p \right] dx = \int_a^b |g(x)| \, \|f^{(n)}\|_p dx + \int_a^b |f(x)| \, \|g^{(n)}\|_p dx$$

$$= \|f^{(n)}\|_p \int_a^b |g(x)| dx + \|g^{(n)}\|_p \int_a^b |f(x)| dx = \|f^{(n)}\|_p \|g\|_1 + \|g^{(n)}\|_p \|f\|_1$$

and, if we apply the inequality (2.9), the first inequality (2.13) follows.

(II) For $p = \infty$, we have

$$\|T_n(x,s)\|_1 = \int_a^b |T_n(x,s)| ds = \frac{1}{n(n+1)(b-a)} \left((x-a)^{n+1} + (b-x)^{n+1} \right) \le \frac{(b-a)^n}{n(n+1)}$$

and, if apply the inequality (2.9), the second inequality (2.13) follows.

(III) For $p = 1$ we have

$$\|T_n(x,s)\|_\infty = \frac{1}{n(b-a)} \max \left\{ (b-x)^n, \, (x-a)^n \right\} \le \frac{(b-a)^{n-1}}{n},$$

and we obtain the third inequality (2.13).

Remark 4. For $n = 1$ the inequalities from the Corollary 2. reduces to

$$|T(f,g)| \le \begin{cases} \dfrac{1}{2(b-a)^2} \left(\dfrac{(b-a)^{q+1}}{q+1} \right)^{1/q} \left[\|f'\|_p \|g\|_1 + \|g'\|_p \|f\|_1 \right], & 1<p<\infty \\ \dfrac{1}{4} \left[\|f'\|_\infty \|g\|_1 + \|g'\|_\infty \|f\|_1 \right], & p=\infty \\ \dfrac{1}{2(b-a)} \left[\|f'\|_1 \|g\|_1 + \|g'\|_1 \|f\|_1 \right., & p=1. \end{cases}$$

For $n = 2$ the inequalities from the Corollary 2. reduces to

$$\left| T(f,g) + \frac{1}{2(b-a)} \int_a^b \left[g(x) f'(x) \frac{(b-x)^2 - (a-x)^2}{2!(b-a)} \right. \right.$$
$$\left. \left. + f(x) g'(x) \frac{(b-x)^2 - (a-x)^2}{2!(b-a)} \right] dx \right|$$
$$\le \begin{cases} \dfrac{1}{4} \dfrac{1}{(b-a)^2} \left(\dfrac{(b-a)^{2q+1}}{2q+1} \right)^{1/q} \left[\|f''\|_p \|g\|_1 + \|g''\|_p \|f\|_1 \right], & 1<p<\infty \\ \dfrac{(b-a)}{12} \left[\|f''\|_\infty \|g\|_1 + \|g''\|_\infty \|f\|_1 \right], & p=\infty \\ \dfrac{1}{4} \left[\|f''\|_1 \|g\|_1 + \|g''\|_1 \|f\|_1 \right], & p=1. \end{cases}$$

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ЗА НЕКОИ НЕРАВЕНСТВА ОД ТИПОТ НА ČEBUŠEV–GRÜSS

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Р е з и м е

Во оваа работа се дадени докази за двете теореми, Теоремата 1 и Теоремата 2, кои ги генерализираат две интегрални неравенства од типот на Čebyšev–Grüs. Со користење на идентитетот на Montgomery ([2] і [3]) се добиени нови неравенства од типот на Čebyšev–Grüs.

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