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BUSY PERIOD AND NUMBER OF CUSTOMERS SERVED DURING THE BUSY PERIOD IN A $\text{Geo}^X/\text{G}_D/1$ QUEUEING SYSTEM

M. Georgieva and V. Bakeva

Abstract

The queueing systems $Geo^X/G_D/1$, i.e. systems with geometrical distributed input stream, group arrivals and discrete service times, are considered in this paper. Using the method of collective marks, the distribution of the busy period and the number of customers serviced during the busy period are determinated.

1. Introduction.

It is known that a geometrical distributed random variable (r.v) has lack-of-memory property, i.e. if $P\{T = k\} = pq^{k-1}$ and X is nonnegative discrete r.v., then the equality

$$P\{T - X \leq n \mid T > X\} = P\{T \leq n\}. \tag{1}$$

holds. An exponential distribution has the same property and therefore we obtained an idea to find some analogues between the exponential and the geometrical distribution. In this paper we use two probabilistic interpretations of a probability generating function (p.g.f.) as follows.

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A. The marking of customers (see Kleinrock [3], Obretenov et al. [4])

Let X be a number of customers which arrive in a queueing system during an interval (for example $[a, b]$) and $0 \leq z \leq 1$. We introduce a marking of customers as follows: each customer can be marked as "red" (independently of others) with probability z and as "blue" with probability $1-z$. Then $P\{X = k\}$ is the probability that exactly k customers arrive in the system during considered interval and $P\{X = k\}z^k$ is the probability that all k arrived customers are "red". By the theorem of total probability, the p.g.f. of X,

$$P_X(z) = \sum_{k=0}^{\infty} P\{X = k\} z^k$$

is the probability that during the considered interval all arrived customers are "red", i.e. no arrival of a "blue" customer.

B. The catastrophe process

Let X be a positive integer valued r.v. which represents the time-interval between two consecutive events from a stream. We consider another stream which is similarly generated by a r.v. $T \sim Geo(1-z)$, $0 < z < 1$ and it is independent from the r.v. X. The event from the stream determined by the r.v. T is called *"catastrophe"*. Then we have

$$P_X(z) = \sum_{k=1}^{\infty} P\{X = k\} z^k = \sum_{k=1}^{\infty} P\{X = k\} P\{T > k\},$$

i.e.

$$P_X(z) = P\{T > X\}.$$

The last relation can be interpreted as follows: *An event from the stream generated by the r.v. X will occur before arrival of a "catastrophe" belonging to the stream generated by the r.v. T.*

These methods are referred together as a method of collective marks.

Remark 1. *The second probabilistic interpretation of a p.g.f. is analogous to the probabilistic interpretation of a Laplace-Stieltjes transform* (see Kleinrock [3], *Obretenov et al.* [4])

In this paper we determinate the p.g.f. of several characteristics of the $Geo^X/G_D/1$ queueing systems. The obtained results are analogous with the same results in continued case for the $M^X/G/1$ system. In Section 2, we give a description of a $Geo^X/G_D/1$ system. In Section 3 and Section 4 we find the functional equations for determination of the p.g.f. of the busy period and the number of customers served during a busy period, correspondingly. In last section we make a conclusion.

2. Description of a $Geo^X/G_D/1$ queueing system

Consider a discrete-time single-server queuing system. The time axis is divided into equal intervals called *slots*. When the customers arrive, they are stored in a buffer (queue). The service of a customer is synchronized to start only at slot boundaries. Without loss of generality we assume that the length of a slot is equal to a unit time. Slots are numbered as nonnegative integers so that the k-th slot correponds to the time-interval $(k-1,k], k=1,2,\ldots$. Let k^- and k^+ be the two time points immediately before and after the time k, correspondingly. Through all our considerations here we assume that:

- a customer completing service in the k-th slot is considered to leave the system sometime in the time (k^-,k);
- a customer whose service starts in the $(k+1)$-th slot commences the service in the time (k,k^+);

We assume that the stream of input moments is geometrical with a parameter p_0, $0<p_0<1$, i.e. the inter-arrival times $T_1, T_2, \ldots$ are independent and identically geometrical distributed r.v.'s defined as follows

$$P\{T=k\}=(1-p_0)^{k-1}p_0, \quad k=1,2,\ldots \tag{2}$$

where T is a generic r.v. We will denote this by $T \sim Geo(p_0)$. The last relation means that during the first $k-1$ slots there are no arrivals and an arrival moment appears just in the following k-th slot, $k=1,2,\ldots$ During each slot, at most one group of customers arrives in the system. The number of customers in a group is a discrete r.v. Z defined by its p.g.f. $\Phi(z)$.

If a group arrives in an empty system the service of a customer from this group starts in the first discrete moment after arrival epoch. If the server is busy, the customers from the group remain in the queue and wait for service according to the queueing discipline. We assume that the service time of a customer is given by positive discrete r.v. X having an arbitrary distribution and p.g.f. $P_X(z)$, $0<z<1$. The described system is usually abbreviated as $Geo(p_0)/G_D/1$. Such types of discrete-time queuing systems with absolutely reliable servers are studied, for instance, by Bruneel and Kim [1], Georgieva [2].

A special case of these systems are systems with ordinary input flow, which means that there is at most one customer per slot.

3. Busy period

Whenever a group arrives in an empty system, it introduces a new busy period which starts at the beginning of the first slot after its arrival instant and terminates in the first moment when the system becomes empty again. It may be noted that the lengths and the positions of the busy period on the time axis are not influenced by the queueing discipline used in the system as long as this discipline is work-conserving (see Kleinrock [3]). Therefore, in order to facilitate the analysis we assume that the last-in-first-out (LIFO) discipline has been adopted. This means that whenever a group leaves the system (after the finishing its service), a customer from the last arrived group is placed into the service. To an arbitrary group, say G_1, we associate a period of time which is referred as a sub-busy period introduced by G_1. This period of time starts at the beginning of the service of the first customer from the group and terminates when the system becomes (for the first time) free from the groups which have arrived after G_1. Evidently, if G_1 is just the group which introduces the busy period then the sub-busy period introduced by G_1 is, actually, the busy period of the system. This means that the above two periods are determinated by the same distribution. The busy period is a positive discrete r.v. B and we denote its p.g.f. as $\Pi(z)$.

Theorem 1. a) *The p.g.f. of the busy period B of the $Geo^X(p_0)/G_D/1$ queueing system satisfies the following functional equation:*

$$\Pi(z) = \Phi\big(P_X(z(q_0 + p_0\Pi(z)))\big) . \tag{3}$$

where $q_0 = 1 - p_0$, $P_X(z)$ is the p.g.f of the service time X and $\Phi(z)$ is the p.g.f. of the number of customers in a group.

b) *If $p_0 EX \cdot EZ < 1$, then the equation* (3) *has a unique solution $\Pi(z)$, which is a p.g.f. of a stochastically bounded r.v.*

c) *If $p_0 EX \cdot EZ < 1$, then the first two moments and variance of B are determinated as follows:*

$$EB = \frac{EX \cdot EZ}{1 - p_0 EX \cdot EZ}, \tag{4}$$

$$EB^2 = \frac{1}{(1 - p_0 EX \cdot EZ)^3}\left((EX)^2 EZ^2 + DX \cdot EZ - p_0^2 (EX)^3 (EZ)^3\right),$$

$$DB = \frac{1}{(1 - p_0 EX \cdot EZ)^3}\left((EX)^2 DZ + DX \cdot EZ + p_0 q_0 (EX)^3 (EZ)^3\right).$$

□

Proof: We have already proved that the busy period of the system and the sub-busy period introduced by a group are determinated by the same distribution. Therefore, we consider the sub-busy period introduced by a group and we are going to find its distribution.

Independently of the service process and the input stream, we introduce a supplementary geometrical stream of "catastrophes" with a parameter $1 - z$. Then, by the probabilistic interpretation of the p.g.f., $\Pi(z)$ is the probability of the event:

$$\left\{ \begin{array}{c} \text{There is not a "catastrophe" during the} \\ \text{sub-busy period introduced by a group} \end{array} \right\}.$$

We refer a group as a "bad" one if there is a "catastrophe" during the busy period introduced by the group. The probability of this event is $1 - \Pi(z)$. The screened stream of "bad" groups from the input geometrical stream with a parameter p_0 is geometrical again, but with a new parameter $p_0(1 - \Pi(z))$.

During the sub-busy period introduced by a group, there will be no "catastrophes", if during the service times of all customers from the group no "catastrophes" and no "bad" customers arrive. The superposition of the streams of "catastrophes" and "bad" groups is a geometrically distributed flow with a parameter

$$1 - z(1 - p_0(1 - \Pi(z))) = 1 - z(q_0 + p_0\Pi(z)).$$

Thus, using the probabilistic interpretation of the p.g.f., $P_X(z(q_0 + p_0\Pi(z))$ is the probability of the event that during the service time of a customer there will be no event of the geometrical stream with parameter $p = 1 - z(q_0 + p_0\Pi(z))$. If the group has exactly k customers, the probability of the event

$$\left\{ \begin{array}{c} \text{There is not event from the superposition of} \\ \text{"catastrophes" and "bad" customers stream} \\ \text{during the service time of all customers} \\ \text{in the group} \end{array} \right\}.$$

is $[P_X(q_0 + p_0\Pi(z))]^k$. But, the number of customers in one group is an arbitrary r.v. Z with set of values $\{1, 2, \ldots\}$, defined by its p.g.f. $\Phi(z)$ and by the of total probability theorem, we have

$$\Pi(z) = \sum_{k=1}^{\infty} (P_X(z(q_0 + p_0\Pi(z))))^k P\{Z = k\} = \Phi(P_X(z(q_0 + p_0\Pi(z))).$$

b) Since $\Phi(P_X(z(q_0 + p_0\Pi(z))))$ is a probability, the unequalities

$$0 \leq \Phi(P_X(z(q_0 + p_0\Pi(z)))) \leq 1. \tag{5}$$

hold. We denote $\Pi(z) = u$ and for fixed z we consider the function:

$$\varphi(u) = \Phi(P_X(z(q_0 + p_0 u))).$$

Since the unequalities (5) are true, φ is a mapping from $[0,1]$ to $[0,1]$.

For the second derivation of $\varphi(u)$ we have $\varphi''(u) \geq 0, \quad \forall u \in [0,1]$. This means that $\varphi'(u)$ does not decrease in the interval $[0,1]$. Moreover, $\varphi'(1) \geq 0$ and we obtain that $0 \leq \varphi'(u) \leq \varphi'(1)$, $\forall u \in [0,1]$ and $\forall z \in [0,1]$. We denote $\varphi'(1) = \alpha$, and by the theorem of Lagrange for average value, we have:

$$|\varphi(u_1) - \varphi(u_2)| = |\varphi'(\xi)|\,|u_1 - u_2| \leq \alpha |u_1 - u_2|, \tag{6}$$

where $u_1,\ u_2 \in [0,1]$ and $u_1 \leq \xi \leq u_2$. If $\alpha < 1$, then the unequality (6) implies that φ is a contraction on $[0,1]$. By the fixed point theorem, for each contraction in a complete metric space, a uniquely determinated fixed point exists. So, there exists $u^* \in [0,1]$ such that $u^* = \varphi(u^*)$.

Since $\varphi(1) = 1$, for $z = 1$, we have $\Pi(1) = 1$, i.e. $u = \Pi(z) = 1$, for $z = 1$. Therefore, since $\varphi'(1) < 1$, we have $p_0 P_X'(1)\Phi'(1) = p_0 EX \cdot EZ < 1$, and $\Pi(z)$ is a p.g.f. of a stochastically bounded r.v.

c) Using the well known relations between the moments of a discrete r.v. and the derivations of its p.g.f.:

$$EB = \Pi'(1), \qquad EB^2 = \Pi''(1) + \Pi'(1).$$

we obtain (4). □

Corollary 1. *Let the input stream in the Geo/G_D/1 system be ordinary, which means that the number of customers in a group is 1. Then,*

a) *the p.g.f. of the busy period B is a solution of the functional equation:*

$$\Pi(z) = P_X(z(q_0 + p_0 \Pi(z))) \tag{7}$$

b) *the equation (7) has unique solution $\Pi(z)$, which is a p.g.f. of a stochastically bounded r.v., if $p_0 EX < 1$,*

c) *the first two moment and the variance of B are determinated as follows:*

$$EB = \frac{EX}{1 - p_0 EX}$$

$$EB^2 = \frac{1}{(1 - p_0 EX)^3}\left((EX)^2 - p_0^2 (EX)^3\right) \tag{8}$$

$$DB = \frac{1}{(1 - p_0 EX)^3}\left(DX + p_0 q_0 (EX)^3\right),$$

if $p_0 EX < 1$. □

4. Number of customers served during a busy period

Evidently, the number of customers served during a busy period is a discrete r.v. C and its distribution is determinated by its p.g.f. $Q(z)$. We state the following

Theorem 1. a) *The function $Q(z)$ satisfies the following functional equation:*

$$Q(z) = \Phi\left(zP_X(q_0 + p_0 Q(z))\right). \tag{9}$$

b) *If $p_0 EX \cdot EZ < 1$, then the equation (9) has a unique solution $Q(z)$, which is a p.g.f. of a stochastically bounded r.v.*

c) *If $p_0 EX \cdot EZ < 1$, then the first two moment and the variance of C are determinated as follows:*

$$EC = \frac{EZ}{1 - p_0 EX \cdot EZ} \tag{10}$$

$$EC^2 = \frac{1}{(1 - p_0 EX \cdot EZ)^3}\left(EZ^2 + p_0^2 DX (EZ)^3 - p_0^2 EX (EZ)^3\right)$$

$$DC = \frac{1}{(1 - p_0 EX \cdot EZ)^3}\left(DZ + p_0^2 DX (EZ)^3 + p_0 q_0 EX (EZ)^3\right).$$

□

Proof: Likewise the length of the busy period, the number of customers served during a busy period does not depend on the queueing discipline. Therefore we can apply LIFO discipline again. Independently of the service process and the input stream we introduce a marking of customers. Each customer can be marked as "red" with probability z or marked as "blue" with probability $1 - z$, $0 < z < 1$. Then, $P\{C = k\}z^k$ is the probability that exactly k customers will be served during a busy period and each od them will be "red", and $Q(z)$ is the probability that during the busy period all served customers are "red". Each group arrived in the system introduce a busy period and in this period the customers from this group and the groups arrived after it will be served. We refer to a group as "dark red" if all customers in the group are "red" and during the busy period introduced by this group all served customers are "red". Otherwise, this group is "light red". Then, $Q(z)$ may be considered as a probability that a group is "dark red", and $1 - Q(z)$ - the probability that a group is "light red". The screened stream of "light red" groups from the input geometrical stream with a parameter p_0 is geometrical with a parameter $p_0(1-Q(z))$. The probability of the event:

$$\left\{\begin{array}{c}\text{There is not a "light red" group during}\\ \text{the service of a customer}\end{array}\right\}$$

is

$$P_X(1-p_0(1-Q(z))) = P_X(q_0+p_0Q(z)).$$

If a group has exactly k customers then $(P_X(q_0+p_0Q(z)))^k z^k$ is the probability of the event:

$$\left\{\begin{array}{c}\text{All customers from a group are "red" and}\\ \text{during their servicing a "light red" group}\\ \text{has not arrived}\end{array}\right\}.$$

But, the number of customers in a group is a r.v. Z with set of values $\{1,2,\ldots\}$, and by the theorem of total probability, we obtain (9). □

Corollary 2. *Let the input stream in the $Geo/G_D/1$ system be ordinary, i.e. let the number of customers in a group be 1. Then,*

a) *the p.g.f. of the number of customer C served during a busy period is a solution of the functional equation:*

$$Q(z) = zP_X\left(q_0+p_0Q(z)\right).$$

b) *the equation* (11) *has unique solution $Q(z)$, which is a p.g.f. of a stochastically bounded r.v., if $p_0EX<1$,*

c) *the first two moment and the variance of C are determinated as follows:*

$$EB = \frac{1}{1-p_0EX}$$

$$EB^2 = \frac{1}{(1-p_0EX)^3}\left(1-p_0^2EX+p_0^2DX\right)$$

$$DB = \frac{1}{(1-p_0EX)^3}\left(p_0^2DX+p_0q_0EX\right),$$

if $p_0EX<1$. □

5. Conclusions

For the queueing $Geo^X/G_D/1$ systems, i.e. systems with geometrical distributed input stream, group arrivals and discrete service times, we use two probabilistic interpretations to determine the p.g.f of the busy period and the number of customers serviced during a busy period. The obtained results are analogous to their Laplace-Stieltjes transforms when $M^X/G/1$ is the considered system (see Obretenov et al. [4]).

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ПЕРИОД НА ЗАФАТЕНОСТ И БРОЈ НА КЛИЕНТИ ОПСЛУЖЕНИ ЗА ВРЕМЕ НА ПЕРИОДОТ НА ЗАФАТЕНОСТ ВО $Geo^X/G_D/1$ СИСТЕМИТЕ ЗА МАСОВНО ОПСЛУЖУВАЊЕ

М. Георгиева и В. Бакева

Резиме

Во овој труд се разгледуваат $Geo^X/G_D/1$ системи т.е. системи со геометриски распределен влезен поток, групно пристигнување на клиентите и дискретно распределено време на опслужување. Со користење на две веројатносни толкувања на генерирачка функција определени се распределбите на периодот на зафатеност и бројот на клиенти опслужени за време на периодот на зафатеност.

The Faculty of the Natural Sciences and Mathematics,
Institute of Informatics, P.O.Box 162, 1000 Skopje,
Republic of Macedonia,

e-mail: {magde,verica}@robig.pmf.ukim.edu.mk