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A RIEMANN-LEBESGUE LEMMA FOR FOURIER-JACOBI COEFFICIENTS

Bujar Xh. Fejzullahu

Abstract

Riemann-Lebesgue Lemma for Fourier-Jacobi coefficients for the class of functions in $L^p_{(\alpha,\beta)}$, $1 < p \leq \infty$, are studied.
In [3], it is proved that

$$\lim_{k\to\infty} \hat{f}_{(\alpha,\beta)}(k) = 0$$

holds for each $f \in L^1_{(\alpha,\beta)}$ if and only if $\alpha \geq \beta > -1$, $\alpha \geq -1/2$.
For $f \in L^p_{(\alpha,\beta)}$, $1 < p \leq \infty$, and $\alpha > -1/2$, $\beta > -1$ Riemann-Lebesgue Lemma is proved.

1. Introduction

For $\alpha > -1$, $\beta > -1$, we denote by $P_n^{(\alpha,\beta)}$ the usual Jacobi polynomials of degree n, orthogonal on $[-1,1]$ with respect to the weight function $\omega_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, (see [5, Chapter IV]). They are normalized such that $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} \sim n^\alpha$, and the $\sim$ sign means that there are positive constants c_1, c_2 such that $c_1 P_n^{(\alpha,\beta)}(1) \leq n^\alpha \leq c_2 P_n^{(\alpha,\beta)}(1)$ holds. We shall often use the notation $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$.
Throughout this paper we denote by $L^p_{(\alpha,\beta)}[-1,1] = L^p_{(\alpha,\beta)}$, $1 \leq p < \infty$, the space of Lebesgue measurable functions with finite norm

$$\|f\|_p = \left(\int_{-1}^{1} |f(x)|^p \omega_{\alpha,\beta}(x)dx \right)^{1/p},$$

and for $p=\infty$, denote by $C[-1,1]=C$, the set of continuous functions on $[-1,1]$ with norm

$$\|f\|_\infty = \max_{-1\le x\le 1} |f(x)|.$$

For $f\in L_{(\alpha,\beta)}$ the Fourier expansion in Jacobi polynomials is

$$f(x) \sim \sum_{n=0}^{\infty} \hat{f}_{(\alpha,\beta)}(n) h_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(x), \tag{1}$$

$$\hat{f}_{(\alpha,\beta)}(n) = \int_{-1}^{1} f(y) R_n^{(\alpha,\beta)}(y)\omega_{\alpha,\beta}(y)dy \tag{2}$$

where (see [3])

$$\begin{aligned} h_n^{(\alpha,\beta)} &= \left(\int_{-1}^{1}\left(R_n^{(\alpha,\beta)}(x)\right)^2 \omega_{\alpha,\beta}(x)dx\right)^{-1} \\ &= \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\beta+1)\Gamma(n+1)\big(\Gamma(\alpha+1)\big)^2} \\ &\sim (n+1)^{2\alpha+1}. \end{aligned}$$

Let $S_n(f;x)$ be the n-th partial sum of the expansion (1)

$$S_n(f;x) = \sum_{k=0}^{n} \hat{f}_{(\alpha,\beta)}(k) h_k^{(\alpha,\beta)} R_k^{(\alpha,\beta)}(x).$$

The classical Riemann-Lebesgue Lemma state (see [1, p.168], [7, (4.4), p.45]):

If $f\in L(-\pi,\pi)$, then

$$\lim_{|k|\to\infty} \int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta = 0$$

In [3], for Fourier-Jacobi coefficients, it is proved:

Let $f\in L_{(\alpha,\beta)}$, then

$$\lim_{k\to\infty} \hat{f}_{(\alpha,\beta)}(k) = 0$$

if and only if $\alpha\ge\beta>-1$, $\alpha\ge -1/2$.

In this paper we will prove this lemma for a class of function $f\in L^p_{(\alpha,\beta)}$, $1<p\le\infty$, $\alpha>-1/2$ and $\beta>-1$.

2. Main results

In ([2], [4]), for $\alpha \geq -1/2$ and $\beta \geq -1/2$, it is proved that

$$\max_{x\in[-1,1]} (1-x)^{\alpha+1/2}(1+x)^{\beta+1/2}\left(n^{1/2}P_n^{(\alpha,\beta)}\right)^2 \leq c,$$

where the positive constant c is independent of n and x.

Hence, for $x \in (-1,1)$, $\alpha \geq -1/2$ and $\beta \geq -1/2$, we have

$$|P_n^{(\alpha,\beta)}(x)| \leq cn^{-1/2}(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}. \tag{3}$$

We are now in position to prove main result.

Lemma 1. *Let the numbers p, α, β be such that $1 < p \leq \infty$;*

$$\begin{array}{lll} \alpha > -1/2, \beta > -1 & & \text{if} \quad 2 \leq p \leq \infty, \\ -1/2 < \alpha < \dfrac{1}{2-p} - \dfrac{3}{2}, & -1 < \beta < \dfrac{1}{2-p} - \dfrac{3}{2} & \text{if} \quad 1 < p < 2. \end{array}$$

Then for $f \in L^p_{(\alpha,\beta)}$, we have

$$\lim_{k\to\infty} \hat{f}_{(\alpha,\beta)}(k) = 0.$$

Proof. As is well known that if $0 < p < r \leq \infty$ and $m(E) < \infty$ then $L^r(E) \subset L^p(E)$ [6, (Theorem 8.2)], so if $f \in L^p_{(\alpha,\beta)} \Rightarrow f \in L_{(\alpha,\beta)}$ $(1 < p \leq \infty)$.

Let $f \in L^p_{(\alpha,\beta)}$, $1 < p \leq \infty$, $\alpha > -1/2$ and $\beta \geq -1/2$. From (2), Hölder's inequality [6, (Theorem 8.6)] and (3), we have

$$|\hat{f}_{(\alpha,\beta)}(n)| \leq c_1 \frac{\|f\|_p}{n^\alpha}\left(\int_{-1}^{1} |P_n^{(\alpha,\beta)}(x)|^q (1-x)^\alpha (1+x)^\beta dx\right)^{1/q}$$

$$\leq c_2 n^{-\alpha-1/2}\|f\|_p \left(\int_{-1}^{1} (1-x)^{\alpha(1-q/2)-q/4}(1+x)^{\beta(1-q/2)-1/4} dx\right)^{1/q},$$

where $q = \frac{p}{p-1}$ for $1 < p < \infty$ and $q = 1$ for $p = \infty$.

The last integral convergent for the $2(2-q)A - q + 4 > 0$, where $A \in \{\alpha, \beta\}$. Hence, $2pA - 4A + 3p - 4 > 0$.

If $2 \le p \le \infty$, then

$$2pA - 4A + 3p - 4 = p(2A+3) - 4A - 4 \ge 2(2A+3) - 4A - 4 = 2.$$

If $1 < p < 2$, then

$$2pA - 4A + 3p - 4 > 0 \Leftrightarrow A < \frac{1}{2-p} - \frac{3}{2}.$$

The prove of the Lemma 1 in the case $f \in L^p_{(\alpha,\beta)}$, $1 < p \le \infty$, $\alpha > -1/2$ and $\beta \ge -1/2$ is completed.

Now we consider case $\alpha > -1/2$ and $\beta \le -1/2$. From [5, (p. 169)], for $x \in (-1, 0]$, $\beta \le -1/2$, we have

$$|P_n^{(\alpha,\beta)}(x)| \le c_3 n^{-1/2}, \tag{4}$$

and for $x \in [0, 1), \alpha > -1/2$, we have

$$|P_n^{(\alpha,\beta)}(x)| \le c_4 n^{-1/2}(1-x)^{-\alpha/2-1/4}, \tag{5}$$

where the positive constants c_3 and c_4 are independent of n and x.

Recall again (2), Hölder's inequality [6, (Theorem 8.6)], (4) and (5), we have

$$\begin{aligned}
|\hat{f}_{(\alpha,\beta)}(n)| &\le c_5 \frac{\|f\|_p}{n^\alpha} \left(\int_{-1}^{1} |P_n^{(\alpha,\beta)}(x)|^q (1-x)^\alpha (1+x)^\beta dx \right)^{1/q} \\
&\le c_5 \frac{\|f\|_p}{n^\alpha} \left((\int_{-1}^{0} + \int_{0}^{1}) |P_n^{(\alpha,\beta)}(x)|^q (1-x)^\alpha (1+x)^\beta dx \right)^{1/q} \\
&\le c_5 n^{-\alpha} \|f\|_p \left(c_6 n^{-q/2} + c_7 n^{-q/2} \int_0^1 (1-x)^{\alpha(1-q/2)-q/4} dx \right)^{1/q} \\
&\le c_8 n^{-\alpha-1/2} \|f\|_p \,.
\end{aligned}$$

□

Corollary 1. *If $f \in L^p_{(\alpha,\beta)}$ and*

$$\begin{array}{lll}
\alpha > -1, \quad \beta > -1 & & \text{if} \quad 2 \le p \le \infty, \\
-1 < \alpha < \dfrac{1}{2-p} - \dfrac{3}{2}, & -1 < \beta < \dfrac{1}{2-p} - \dfrac{3}{2} & \text{if} \quad 1 < p < 2
\end{array}$$

then,

$$\lim_{n \to \infty} \int_{-1}^{1} f(x) P_n^{(\alpha,\beta)}(x)(1-x)^\alpha (1+x)^\beta dx = 0.$$

Proof. We will take the case $-1 < \alpha \leq -1/2$, $-1 < \beta \leq -1/2$, since the case $-1 < \alpha \leq -1/2$, $\beta > -1/2$ is similarly case as $-1 < \beta \leq -1/2$, $\alpha > -1/2$.

From the Hölder's inequality [6, (Theorem 8.6)] and [5, (p. 169)] we have

$$\left|\int_{-1}^{1} f(x)P_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx\right|$$

$$\leq c_1\|f\|_p \left((\int_{-1}^{0} + \int_{0}^{1}) |P_n^{(\alpha,\beta)}(x)|^q (1-x)^{\alpha}(1+x)^{\beta}dx \right)^{1/q} \leq c_2 n^{-1/2}\|f\|_p.$$

□

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RIEMANN-LEBESGUE-ОВА ЛЕМА НА КОЕФИЦИЕНТИТЕ НА FOURIER-JACOBI

Bujar Xh. Fejzullahu

Р е з и м е

Во овој труд ја докажавме Riemann-Lebesgue-овата лема за коефициентите на Fourier-Jacobi за класа функции во $L^p_{(\alpha,\beta)}$ $(1 < p \leq \infty)$ за соодветни α и β.

Faculty of Mathematics and Sciences
University of Prishtina
Nëna Terezë 5
10000 Prishtinë Kosovë

e-mail: bujarfej@uni-pr.edu