SOME RESULTS ON GENERALIZATIONS OF FOX'S H-FUNCTIONS

By

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In the course of an investigation an attempt has been made by several workers for the unification and extension of certain recent results in the theory of special function. The author evaluates here some finite integrals for the extended H-functions in two arguments with Gauss's hypergeometric functions which generalize a number of particular interesting and useful results scattered throughout literature and also the double-integral analogues for one of the results has been derived.

§. 1. **Introduction:**

An extension of Fox's H-function [(5), p. 408] has been recently introduced by Agarwal and Mathur [(1). p. 536] in the form

$$(1.\ 1)\qquad H\begin{bmatrix} x \\ y \end{bmatrix} \equiv H^{n,\ \nu_1,\ \nu_2,\ m_1, m_2}_{p,\ [t:t'],\ s,\ [q:q']}\left[\begin{array}{c|c} x & \{(\varepsilon_p,\ e_p)\} \\ & \{(\gamma_t,\ c_t)\};\ \{(\gamma_{t'},\ c_{t'})\} \\ y & \{(\delta_s,\ d_s)\} \\ & \{(\beta_q,\ b_q)\};\ \{(\beta_{q'},\ b_{q'})\} \end{array}\right] =$$

$$= \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Phi(\xi+\eta)\,\psi(\xi,\ \eta)\,x^{\xi}\ y^{\eta}\ d\xi\ d\eta$$

where

(i) $o \leqslant n \leqslant p,\quad o \leqslant \nu_1 \leqslant t,\quad o \leqslant \nu_2 \leqslant t',\quad o \leqslant m_1 \leqslant q,\quad o \leqslant m_2 \leqslant q';$

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(ii) $\{(\varepsilon_p, e_p)\}$ represents the set of p-parameters:

$(\varepsilon_1, e_1), \ldots\ldots, (\varepsilon_n, e_n); (\varepsilon_{n+1}, e_{n+1}), \ldots\ldots, (\varepsilon_p, e_p)$

and similarly for $\{(\gamma_t, c_t)\}$, $\{(\gamma_t', c_t',)\}$ and so on;

(iii) all e's, c's etc. are positive integers:

(iv)
$$\begin{cases} 2(n+\nu_1+m_1) > p+s+t+q \\ 2(n+\nu_2+m_2) > p+s+t'+q' \\ |\arg(x)| < [n+\nu_1+m_1-1/2(c+s+t+q)]\pi, \\ |\arg(y)| < [n+\nu_2+m_2-1/2(p+s+t'+q')]\pi; \end{cases}$$

(v) L_1 and L_2 are suitable contours;

and

(vi)
$$\begin{cases} \Phi(\xi+\eta) = \dfrac{\prod_{j=1}^{n}\Gamma(1-\varepsilon_j+e_j\xi+e_j\eta)}{\prod_{j=n+1}^{b}\Gamma(\varepsilon_j-e_j\xi-e_j\eta)\prod_{j=1}^{s}\Gamma(\delta_j+d_j\xi+d_j\eta)} \\ \psi(\xi,\eta) = \\ = \dfrac{\prod_{j=1}^{\nu_1}\Gamma(\gamma_j+c_j\xi)\prod_{j=1}^{m_1}\Gamma(\beta_j-b_j\xi)\prod_{j=1}^{\nu}\Gamma(\gamma_j'+c_j'\eta)\prod_{j=1}^{m_2}\Gamma(\beta_j'-b_j'\eta)}{\prod_{j=\nu_1+1}^{t}\Gamma(1-\gamma_j-c_j\xi)\prod_{j=m_1+1}^{q}\Gamma(1-\beta_j+b_j\xi)\prod_{j=\nu_2+1}^{t'}\Gamma(1-\gamma_j'-c_j'\eta)\prod_{j=m_2+1}^{q'}\Gamma(1-\beta_j'+b_j'\eta)} \end{cases}$$

This paper deals with the evaluation of some finite integrals involving the product of the extended H-functions of two variables and Gauss's hypergeometric functions by method based on interchanging the order of integration, term-by-term integration, summing the inner $_3F_2(+1)$ with the help of the theorems due to Saalschiitz, Watson and Whipple and then using Gauss's multiptication formula etc. Tese formulae lead to several generalizations of the results recently obtained by Mac Robert [(6)], Verma [(11)], Patnak [(8)], Denis [(3)] and Shah [(9)] and one of the double-integral analogues of the results has been also illustrated.

§. 2. The General results

In this section, we have evluated the three main integrals on generalised Fox's H-functions and Gauss's hyrergeometric functions. First Integral:

$$(2.\ 1)\quad \int_0^1 u^{\rho-1}(1-u)^{\sigma-\rho-1}{}_2F_1\begin{pmatrix}\lambda, -k & ; u\\ 1+\lambda+\rho-\sigma k\end{pmatrix} H\begin{bmatrix} x\,u^l\\ y\,u^l\end{bmatrix} du =$$

$$= \frac{\Gamma(\sigma-\rho+k)l^{\rho-\sigma}}{(\sigma-\lambda-\rho)_k} H^{n+2l,\ \nu_1,\ \nu_2\ m_1\ m_2}_{p+2l,\ [t:t'],s+2l,\ [q:q']}\left[\begin{matrix}x\\ y\end{matrix}\left|\begin{matrix}\Delta(l, 1-\rho),\ \Delta(l, \lambda-\sigma-k+1),\\ \{(\varepsilon_p, e_p)\}\\ \{(\gamma_t, c_t)\};\ \{(\gamma'_{t'}, c'_{t'})\}\\ \{(\delta_s\ d_s)\},\ \Delta(l, \sigma-\beta),\ \Delta(l, \sigma+k)\\ \{(\beta_q, b_q)\};\ \{(\beta_q', b'_q')\}\end{matrix}\right.\right]$$

provided

$Re(\sigma) > Re(\rho)$, l iz a positive integer > 0, and

$$\text{(i)}\quad \begin{cases} p+q+s+t < 2(m_1+\nu_1+n),\\ p+q'+s+t' < 2(m_2+\nu_2+n),\\ |\arg(x)| < [m_1+\nu_1+n-1/2(p+q+s+t)]\pi,\\ |\arg(y)| < [m_2+\nu_2+n-1/2(p+q'+s+t')]\pi,\\ Re\left[\rho+l\left(\frac{\beta_j}{b_j}\right)+l\left(\frac{\beta_i'}{b_i'}\right)\right] > 0,\ j=1,\ 2,\dots,m_1;\ i=1,\ 2,\dots,\ m_2\end{cases}$$

or

$$\text{(ii)}\quad \begin{cases} p+t < s+q,\ p+t' < s+q',\\ \text{or else } p+t = s+q,\ p+t' = s+q' \text{ with } |x|<1,\ |y|<1,\\ Re\left[\rho+l\left(\frac{\beta_j}{b_j}\right)+l\left(\frac{\beta_i'}{b_i'}\right)\right] > 0,\ j=1,\ 2,\dots,m_1;\ i=1,\ 2,\dots,m_2.\end{cases}$$

The symbol $\Delta(m, n)$ denotes the set of m- parameters;

$$\frac{n}{m}, \frac{n+1}{m}, \dots\dots\dots, \frac{n+m-1}{m}.$$

Second Integral:

$$(2.\ 2)\quad \int_0^1 u^{\rho-1}(1-u)^{\rho-l}{}_2F_1\begin{bmatrix}\mu, \lambda & ; u\\ \frac{1}{2}(\mu+\lambda+1)\end{bmatrix} H\begin{bmatrix} x\,u^l(1-n)^l\\ y\,u^l(1-u)^l\end{bmatrix} du =$$

$$= \frac{\pi}{\sqrt{l}\ 2^{2\rho-1}}\Gamma\begin{bmatrix}\frac{1}{2}(\mu+\lambda+1);\\ \frac{1}{2}(\mu+1),\ \frac{1}{2}(\lambda+1)\end{bmatrix} H^{n+2l,\ \nu_1,\ \nu_2,\ m_1\ m_2}_{p+2l,\ [t:t'],\ s+2l,\ [q:q']}$$

$$\left[\begin{matrix}\frac{x}{2^{2l}}\\ \\ \\ \frac{y}{2^{2l}}\end{matrix}\left|\begin{matrix}\Delta(l,\,1-\rho),\ \Delta\left(l,-\rho+\frac{1}{2}\mu+\frac{1}{2}\lambda+\frac{1}{2}\right),\ \{(\varepsilon_p,\,e_p)\}\\ \{(\gamma_t,\,c_t)\};\ \{(\gamma'_{t'},\,c'_{t'})\}\\ \{(\delta_s,\,d_s)\},\ \Delta\left(l,\,\rho+\frac{1}{2}-\frac{1}{2}\mu\right),\ \Delta\left(l,\,\rho+\frac{1}{2}-\frac{1}{2}\lambda\right)\\ \{(\beta_q,\,b_q)\};\ \{(\beta'_{q'},\,b'_{q'})\}\end{matrix}\right.\right]$$

where l is a positiv integer > 0, $Re(\mu+\lambda) > -1$,and

$$\text{(i)}\quad\begin{cases} p+q+s+t<2(m_1+\nu_1+n, \ |\arg(x)|<[m_1+\nu_1+\\ \qquad\qquad +n-1/2(p+q+s+t)]\pi,\\ p+q'+s+t'<2(m_2+\nu_2+n), \ |\arg(y)|<[m_2+\nu_2+\\ \qquad\qquad +n-1/2(p+q'+s+t')]\pi,\\ Re\left[\rho+l\left(\frac{\beta_j}{b_j}\right)+l\left(\frac{\beta_i}{b_i}\right)\right]>0,\ j=1,\,2,\ldots,m_1;\ i=1,\,2,\ldots,m_2 \end{cases}$$

or

$$p+t<s+q,\quad p+t'<s+q'.$$

(ii) or else $p+t=s+q$, $p+t'=s+q'$ with $|x|<1$, $|y|<1$,

$$Re\left[\rho+l\left(\frac{\beta_j}{b_j}\right)+l\left(\frac{\beta_i'}{b_i'}\right)\right]>0,\quad j=1,2,\ldots,m_1;\ i=1,2,\ldots,m_2.$$

The notation $\Gamma\left[\begin{matrix}a_p\ ;\\ a_q\end{matrix}\right]$ stands for the product of, the type

$$\frac{\Gamma(a_1)\,\Gamma(a_2)\ldots\ldots\Gamma(a_p)}{\Gamma(b_1)\,\Gamma(b_2)\ldots\ldots\Gamma(b_q)}.$$

Third Integral:

$$(2.\,3)\quad \int_0^1 u^{\rho-1}(1-u)^{\rho-\sigma}\,{}_2F_1\left(\begin{matrix}\alpha,\ 1-\alpha\\ \sigma\end{matrix};\,u\right)H\left[\begin{matrix}x\,u^l\,(1-u)^l\\ y\,u^l\,(1-u)^l\end{matrix}\right]du=$$

$$=\frac{\pi}{\sqrt{l}\,2^{2\rho-1}}\,\Gamma\left[\begin{matrix}\sigma\,;\\ \frac{1}{2}(\alpha+\sigma),\,\frac{1}{2}(\sigma-\alpha+1)\end{matrix}\right]H^{n+2l,\ \nu_1\ \nu_2,\ m_1,\ m_2}_{p+2l,\ [t:t'],\ s+2l,\ [q:q']}$$

$$\left[\begin{array}{c|c} \frac{x}{2^{2l}} & \Delta(l,\ 1-\rho),\ \Delta(l, -\rho-\sigma),\ \{\varepsilon_p,\ e_p)\} \\ & \{(\gamma_t,\ q_t)\}\ ;\ \{(\gamma'_{t'},\ c'_{t'})\} \\ & \{(\delta_s\ d_s)\},\ \Delta\left(l,\ \rho+\frac{1}{2}\alpha-\frac{1}{2}\sigma+\frac{1}{2}\right),\ \Delta\left(l, \rho+1-\frac{1}{2}\alpha-\frac{1}{2}\sigma\right) \\ \frac{y}{2^{2l}} & \{(\beta_q,\ b_q)\}\ ;\ \{(\beta'_{q'},\ b'_{q'})\} \end{array}\right]$$

valid for l a positive integer > 0, $Re(\sigma) > 0$ and

$$\text{(i)}\quad \begin{cases} p+q+s+t < 2(m_1+\nu_1+n), \\ |\arg(x)| < \left[m_1+\nu_1+n-\frac{1}{2}(p+q+s+t)\right]\pi, \\ p+q'+s+t' < 2(m_2+\nu_2+n), \\ |\arg(y)| < \left[m_2+\nu_2+n-\frac{1}{2}(p+q'+s+t')\right]\pi \\ Re\left[\rho+l\left(\frac{\beta_j}{b_j}\right)+l\left(\frac{\beta_i'}{b_i'}\right)\right] > Re(\sigma) > -1 \\ j=1,2,\ldots,m_1;\quad i=1,2,\ldots,m_2, \end{cases}$$

or

$$\text{(ii)}\quad \begin{cases} p+t < s+q,\quad p+t' < s+q', \\ \text{or else } y+t=s+q,\quad p+t'=s+q' \text{ with } |x|<1,\ |y|<1, \\ Re\left[\rho+l\left(\frac{\beta_j}{b_j}\right)+l\left(\frac{\beta_i'}{b_i'}\right)\right] > Re(\sigma)-1\ \ 0, \\ \qquad j=1,2,\ldots,m_1;\quad i=1,2,\ldots,m_2. \end{cases}$$

Proofs:

(A) To prove (2. 1), we substitute the contour integral (1. 1) for $H\begin{bmatrix} x\ u^l \\ y\ u^l \end{bmatrix}$ in the integrand, interchange the order of integration which can readily be justified by de la Vallée Poussin's theorem [(2), p, 504] under the condition stated in (2. 1) earlier, and then interpret the inner u-integral by term-by temm integration or making use of the known formula [(6), p. 850, (12)]:

$$\int_0^1 x^{u-1}(1-x)^{v-1}\,{}_pF_q\left(\begin{matrix}a_p\\ b_q\end{matrix};\ \alpha\,x\right)dx = \frac{\Gamma(u)\,\Gamma(v)}{\Gamma(u+v)}\,{}_{p+1}F_{q+1}\left(\begin{matrix}a_p, u\\ b_q,\ u+v\end{matrix};\ \alpha\right)$$

where $Re(u) >, 0$ $Re(v) > 0$, $p \leqslant q+1$, if $p = q+1$, then $|\alpha| < 1$, we obtain

$$(2.\ 4) \quad \Gamma(\sigma-\rho)\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Phi(\xi+\eta)\psi(\xi,\ \eta)\Gamma\begin{bmatrix}\rho+l\xi+\eta; \\ \sigma+l\xi+l\eta\end{bmatrix}$$

$$ {}_3F_2\begin{pmatrix}\lambda, -k,\ \rho+l\xi+l\eta; & 1 \\ 1+\lambda+\rho-\sigma-k,\ \sigma+l\xi+l\eta\end{pmatrix} x^{\xi}\ y^{\eta}\,d\xi\,d\eta.$$

Now summing up the inner ${}_3F_2(+1)$ in (2. 4) with the application of Saalschiitz's theorem [(4), p. 188, (3)]:

$${}_3F_2\begin{pmatrix} -n,\ a,\ b\ : \\ c,\ 1+a+b-c-n\end{pmatrix} = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n},\ n=0,\ 1,\ 2,\ldots,$$

and by virtue of the multiplication theorem [(4), p. 4, (11)]:

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}(1-m)}\ m^{mz-\frac{1}{2}}\prod_{i=0}^{m-1}\Gamma\left(z+\frac{i}{m}\right),\ m=2,\ 3,\ 4,\ldots,$$

we get $\quad \dfrac{\Gamma(\sigma-\rho+k)l^{\rho-\sigma}}{(\sigma-\lambda-\rho)_k}\dfrac{1}{(2\pi i)^2}\displaystyle\int_{L_1}\int_{L_2}\Phi(\xi+\eta)\psi(\xi,\eta)$

$$(2.\ 5)\quad \frac{\prod_{i=0}^{l-1}\Gamma\left(\frac{\rho+i}{l}+\xi+\eta\right)\prod_{i=0}^{l-1}\Gamma\left(\frac{\sigma-\lambda+k+i}{l}+\xi+\eta\right)x^{\xi}\ y^{\eta}}{\prod_{i=0}^{l-1}\Gamma\left(\frac{\sigma-\lambda+i}{l}+\xi+\eta\right)\prod_{i=0}^{l-1}\Gamma\left(\frac{\sigma+k+i}{l}+\xi+\eta\right)}\,d\xi\ d\eta.$$

The contour L_1 in the ξ—plane here is a straight line along the imaginary axis extending from $-i\infty$ to $+i\infty$ with indentation, if necessary to the ensure that the poles of $\Gamma(\beta_j-b_j\xi)$, $j=1,\ 2,\ldots,\ m_1$ lie to the right of it and the poles of $\Gamma(\gamma_j+c_j\xi)$, $j=1,\ 2,\ldots,\ \nu_1$ and $\Gamma(1-\varepsilon_j+$ $+e_j\xi+e_j\eta)$, $j=1,\ 2,\ldots,\ n$, $\Gamma\left(\dfrac{\rho+i}{l}+\xi+\eta\right)$, $\Gamma\left(\dfrac{\sigma-\lambda+k+i}{l}+\xi+\eta\right)$, $\{i=0,\ 1,\ldots,\ (l-1)\}$ are to the left of the contour.

Similarly the contour L_2 in the η—plane runs from $-i\infty$ *to* $+i\infty$ with loops, if necessary to ensure that the poles of $\Gamma(\beta_j'-b_j'\eta)$, $j=1,\ 2,$ $\ldots, m_2$, lie to the right and $\Gamma(\gamma_j'+c_j'\eta)$, $j=1,\ 2,\ldots,\ \nu_2$, $\Gamma(1-\varepsilon_j+e_j\xi+$ $+e_j\eta)$, $j=1,\ 2,\ldots,n$, $\Gamma\left(\dfrac{\rho+i}{l}+\xi+\eta\right)\Gamma\left(\dfrac{\sigma-\lambda+k+i}{l}+\xi+\eta\right)$, $\{i=0,\ 1,\ 2,\ldots,\ (l-1)\}$ are to the left of the contour.

Therefore. the value of integral (2. 1) follows immediately from (2. 5) in view of (1. 1).

(B) The proof of integral (2. 2) is based on the same lines as above in (2. 1) and summing up the inner ${}_3F_2(+1)$ by Watson's theorem [(4), p. 189, (6)]:

$$ {}_3F_2\left[\begin{matrix} a,\ b,\ c & ; \\ \frac{1}{2}(a+b+1),\ 2c \end{matrix}\right] = $$

$$ = \sqrt{\pi}\,\Gamma\left[\begin{matrix} c+\frac{1}{2},\ \frac{1}{2}(a+b+1),\ c+\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}b; \\ \frac{1}{2}(a+1),\ \frac{1}{2}(b+1),\ c+\frac{1}{2}-\frac{1}{2}a,\ c+\frac{1}{2}-\frac{1}{2}b \end{matrix}\right] $$

where $Re(c) > -1/2$, $Re(a+b) > -1$, $Re(2c-a-b) > -1$.

(*C*) The integral (2. 3) can easily be evaluated similar to (2. 1) and summing the inner ${}_3F_2(+1)$ by virtue of Whipple's theorem [(4), p. 189, (7)]:

$$ {}_3F_2\left(\begin{matrix} a,\ 1-a,\ c; \\ f,\ 2c+1-f \end{matrix}\right) = $$

$$ = \pi\, 2^{1-2c}\,\Gamma\left[\begin{matrix} f,\ 2c+1-f\ ; \\ c+\frac{1}{2}a+\frac{1}{2}-\frac{1}{2}f,\ \frac{1}{2}a+\frac{1}{2}f,\ c+1-\frac{1}{2}a-\frac{1}{2}f,\ \frac{1}{2}-\frac{1}{2}a+\frac{1}{2}f \end{matrix}\right] $$

where $Re(f) > 0$, and $Re\ (2c-f) > -1$.

§ 3. Applications

Particular cases:-

If we take all $e's$, $c's$, $c's$, $d's$, $b's$ and $b''s$ equal to unity in (2, 1), we obtain the known recent result of Denis [(3), p. 3 (3. 1)].

(ii) For $p = s = n = 0$, the double integral in (1. 1) breaks up into the product of two Fox's H-functions:

$$ H_{t,\ q}^{m_1,\ \nu_1}\left[x\left|\begin{matrix}\{(1-\gamma_t,\ c_t)\} \\ \{(\beta_q,\ b_q)\}\end{matrix}\right.\right] H_{t',\ q'}^{m_2,\ \nu_2}\left[y\left|\begin{matrix}\{(1-\gamma'_{t'},\ c'_{t'})\} \\ \{(\beta'_{q'},\ b'_{q'})\}\end{matrix}\right.\right].\ (q \geqslant t,\ q' \geqslant t') $$

and from (2. 1) we thus obtain

$$(3.\ 1)\ \int_0^1 u^{\rho-1}(1-u)^{\sigma-\rho-1}\,{}_3F_2\left(\begin{matrix}\lambda, -k & ; u\\ 1+\lambda+\rho-\sigma-k\end{matrix}\right)$$

$$H_{t,\ \ q}^{m_1,\ \nu_1}\left[x\,u^l\left|\begin{matrix}\{\gamma_t,\ c_t)\}\\ \{\beta_q,\ b_q)\}\end{matrix}\right.\right] H_{t',\ \ q'}^{m_2,\ \nu_2}\left[y\,u^l\left|\begin{matrix}\{\gamma'_{t'}, c'_{t'})\}\\ \{\beta'_{q'}\ b'_{q'}\}\end{matrix}\right.\right] du =$$

$$= \frac{\Gamma(\sigma-\rho+k)\, l^{\rho-\sigma}}{(\sigma-\lambda-\rho)_k}\, H_{2l,\ [t:t'],\ 2l,\ [q:q']}^{2l,\ \nu_1,\ \nu_2,\ m_1,\ m_2}$$

$$\left[\begin{matrix}x\\ \\ y\\ \end{matrix}\left|\begin{matrix}\Delta(l,\ 1-\rho),\ \Delta(l,\ \lambda-\sigma-k+1)\\ \{(\gamma_t, c_t)\};\ \{(\gamma'_{t'},\ c'_{t'})\}\\ \Delta(l,\ \sigma-\lambda),\ \Delta(l,\ \sigma+k)\\ \{(\beta_q,\ b_q)\};\ \{(\beta'_{q'},\ b'_{q'})\}\end{matrix}\right.\right]$$

which holds under the same conditions as in (2. 1) with $p=s=n=0$.

(iii) By setting the parameters suitably in (1. 1), we have

$$\lim_{y\to 0} H_{p,\ [t:0],\ s,\ [q:1]}^{p,\ \nu_1,\ 0,\ m_1,\ 1}\left[\begin{matrix}x\\ \\ y\end{matrix}\left|\begin{matrix}\{(\varepsilon_p,\ e_p\}\\ \{(\gamma_t,\ c_t)\};-\\ \{(\delta_s,\ d_s)\}\\ \{(\beta_q, b_q)\};\ 0\end{matrix}\right.\right] =$$

$$= H_{p+t,\ s+q}^{m_1,\ p+\nu_1}\left[x\left|\begin{matrix}\{(\varepsilon_p\ e_p)\},\ \{(1-\gamma_t,\ c_t)\}\\ \{(\beta_q,\ b_q)\},\ \{1-\delta_s,\ d_s)\}\end{matrix}\right.\right]$$

the special case $p=t'=n=\nu_2=m_2-1=q'-1=0$ and then replacing $c+\nu_1$ by ν_1, $p+t$ by t and $s+q$ by q respectively of (2. 1) yields the formula

$$(3.\ 2)\ \int_0^1 u^{\rho-1}(1-u)^{\sigma-\rho-1}\,{}_2F_1\left(\begin{matrix}\lambda, -k & ; u\\ 1+\lambda+\rho-\sigma-k\end{matrix}\right) H_{t,\ q}^{m_1,\nu_1}\left[x\,u^l\left|\begin{matrix}\{(\gamma_t,\ c_t)\}\\ \{(\beta_q, b_q)\}\end{matrix}\right.\right] du =$$

$$= \frac{\Gamma(\sigma-\rho+k)\, l^{\rho-\sigma}}{(\sigma-\lambda-\rho)_k}\, H_{t+2l,\ q+2l}^{m_1,\ \nu_1+2l}\left[x\left|\begin{matrix}\Delta(l, 1-\rho), \Delta(l, \lambda-\sigma-k+1), \{(\gamma_t, c_t)\}\\ \{(\beta_q, b_q)\}, \nabla(l, \sigma-\lambda), \nabla(l, \sigma+k)\end{matrix}\right.\right]$$

where, fo convergence, $Re(\sigma) > Re(\rho)$, l is a positive integer >0 and

$$\text{(i)}\quad \begin{cases} 2(m_1+\nu_1) > t+q, \\ |\arg(x)| < \left[m_1+\nu_1-\frac{1}{3}(t+q)\right]\pi, \\ Re\left[\rho + l\left(\frac{\beta_j}{b_j}\right)\right] > 0,\ j=1,\ 3, \ldots, m_1 \end{cases}$$

or

$$\text{(ii)}\quad \begin{cases} t > q, \\ \text{or else } t = q \text{ with } |x| < 1, \\ Re\left[\rho + l\left(\frac{\beta_j}{b_j}\right)\right] > 0,\ j=1, \ldots, m_1, \end{cases}$$

while $\nabla\ (m,\ n)$ will stand for the sequence

$$1-\frac{n}{m},\ 1-\frac{n+1}{m},\ 1-\frac{n+2}{m}, \ldots\ldots,\ 1-\frac{n+m-1}{m}.$$

As the H-functions are generalizations of Meijer G-functions on specializing the parameters we can obtain many results, some of which are known and others are believed to be new, interesting andalso useful in the analysis of many problems of mathematics, both pure and applied, and in mathematical physics.

Similar results can be obtained from (2. 2) and (2. 3) which include extensions of many particular cases giyen by Mac Robert [(7)] Denis [(3), p. 3, (3. 2) and (3. 3)], Pathak [(8), pp. 585—86 (1)—(2)] and Shah [(9), p. 5, (2. 1)—(2. 2)].

We remark in passing that on th same procedure of (2. 1), we can still evaluate the integral of the form

$$(3.\ 3)\quad \int_0^1 u^{\sigma-1}(1-u)^{\rho-\sigma-1} F_m(u)$$

$$H\begin{matrix} n,\ \nu_1,\ \nu_2,\ m_1,\ m_2 \\ p,\ [t:t'],\ s,\ [q:q'] \end{matrix}\left[\begin{matrix} x u^k (1-u)^h \\ y u^l (1-u)^r \end{matrix}\left|\begin{matrix} \{(\varepsilon_p\ e_p)\} \\ \{(\gamma_t, c_t)\};\ \{(\gamma'_{t_1},\ c'_{t_1})\} \\ \{(\delta_s,\ d_s)\} \\ \{(\beta_q,\ b_q)\};\ \{(\beta'_{q_1},\ b'_{q_1})\} \end{matrix}\right.\right] du$$

for either $k=h=0$, $l=r$ or $l=r=0$, $k=h$; or $k=h=r=0$ or $k=r=l=0$ and where

$$F_m(u)=u^{(\lambda-1)m}\,{}_{P+\lambda}F_Q\left[\begin{matrix}\Delta(\lambda,-m),\ A_p\\ B_Q\end{matrix};\ \mu u^{\xi}\right]$$

is the generalized hypergeometric polynomial introduced by the author in his recent paper [(10), p. 79, (2. 1)] which yields many known polynomials with special values of the parameters and argument μu^{ξ}. Thus by particular choice of parameters in (3. 3), we can easily evaluate its value.

§ 4 The double-integral analogues

In this section we have derived in a straight forward manner the double-integral analogues for the extended H-function in two arguments and Gauss's hypergeometric function by the method of the preceding section.

The double-integral analogues is:

$$(4.1)\quad \int_0^1\int_0^1 u^{\rho-1}(1-u)^{\sigma-\rho-1}\,v^{\sigma-\rho-1}(1-v)^{\alpha-\mu-1}$$

$${}_2F_1\left(\begin{matrix}\lambda,-k\\ 1+\lambda+\rho-\sigma-k\end{matrix};u\right){}_2F_1\left(\begin{matrix}\beta,-r\\ 1+\beta+\mu-\alpha-r\end{matrix};v\right)$$

$$H^{n,\ \nu_1,\ \nu_2,\ m_1,\ m_2}_{p,\ [t:t'],\ s,\ [q:q']}\left[\begin{matrix}xu^l\\ yv^h\end{matrix}\left|\begin{matrix}\{(\varepsilon_p,\ e_p)\}\\ \{(\gamma_t,\ c_t)\};\ \{(\gamma'_{t'},\ c'_{t'})\}\\ \{(\delta_s,\ d_s)\}\\ \{(\beta_q,\ b_q)\};\ \{(\beta'_{q'},\ b'_{q'})\}\end{matrix}\right.\right]du\,dv=$$

$$=\frac{l^{\rho-\sigma}h^{\mu-\alpha}\,\Gamma(\sigma-\rho+k)\,\Gamma(\alpha-\mu+r)}{(\sigma-\lambda-\rho)_k\,(\alpha-\beta-\mu)_r}$$

$$H^{n,\ \nu_1+2l,\ \nu_2+2h,\ m_1,\ m_2}_{p,\ [t+2l:\ t'+2h],\ s,\ [q+2l:\ q'+2h]}$$

$$\left[\begin{matrix}x\\ y\end{matrix}\left|\begin{matrix}\{(\varepsilon_p,\ e_p)\}\\ \Delta(l,\ \rho),\ \Delta(l,\ \sigma-\lambda+k),\ \{(\gamma_t,c_t)\};\ \Delta(h,\ \mu),\ \Delta(h,\ \alpha-\beta+r),\ \{\gamma'_{t'},\ c'_{t'})\}\\ \{(\delta_s,\ d_s)\}\\ \{(\beta_q,\ b_q)\},\ \Delta(l,\ 1-\sigma+\lambda),\ \Delta(l,\ 1-\sigma-k);\ \{(\beta'_q,\ b'_{q'})\},\ \Delta(h,\ 1-\\ -\alpha+\beta),\ \Delta(h,\ 1-\alpha-r)\end{matrix}\right.\right]$$

where l and h are positive integers >0,

$Re(\sigma) > Re(\rho)$, $Re(\alpha) > Re(\mu)$ and set of the conditions of the validity:

$$\text{(i)}\quad \begin{cases} p+q+s+t < 2(m_1+\nu_1+n) \\ p+q'+s+t' < 2(m_2+\nu_2+n), \\ |\arg(x)| < [m_1+\nu_1+n-\frac{1}{2}(p+q+s+t)]\pi \\ |\arg(y)| < [m_2+\nu_2+n-\frac{1}{2}(p+q'+s+t')]\pi, \\ Re\left[\rho+l\left(\frac{\beta_j}{b_j}\right)\right] > 0, \quad j=1,2,\ldots,m_1, \\ Re\left[\mu+h\left(\frac{\beta_i'}{b_i'}\right)\right] > 0, \quad i=1,2,\ldots,m_2 \end{cases}$$

or

$$\text{(ii)}\quad \begin{cases} p+t < s+q, \quad p+t' < s+q' \\ \text{or else } p+t = s+q, \quad p+t' = s+q' \text{ with } |x|<1, \quad |y|<1, \\ Re\left[\rho+l\left(\frac{\beta_j}{b_j}\right)\right] > 0, \quad i=1,2,\ldots,m_1 \\ Re\left[\mu+h\left(\frac{\beta_i'}{b_i'}\right)\right] > 0, \quad i=1,2,\ldots,m_2 \end{cases}$$

Similar results can also be obtained from (2. 2) and (2. 3).

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