

## EXPRESSIONS OF LAGUERRE POLYNOMIALS THROUGH BERNOULLI POLYNOMIALS

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### Abstract

The aim of this paper is to obtain an expansion formula of Laguerre polynomials through Bernoulli polynomials.

### 1. Introduction

Laguerre polynomials  $L_n^{(\alpha)}(x)$  are defined by

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} {}_1F_1(-n; 1+\alpha; x), \quad n \geq 0 \quad (1)$$

where

$${}_1F_1(a; b; x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(b)_n n!}$$

is the confluent hypergeometric function and

$$\begin{aligned} (\alpha)_n &= \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1), & n > 1 \\ (\alpha)_0 &= 1, & \alpha \neq 0 \end{aligned}$$

is the factorial function [1]. It is a special case of the generalized hypergeometric function  ${}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; x)$  used later.

From (1) it follows immediately that

$$L_n^{(\alpha)}(x) = \sum_{k=0}^n \frac{(-1)^k (1+\alpha)_n x^k}{k!(n-k)!(1+\alpha)_k}$$

Bernoulli polynomials  $B_n(x)$  are defined by

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (2)$$

from where we have

$$B_n(x) = \sum_{r=0}^n \binom{n}{r} B_r x^{n-r}. \quad (3)$$

$B_r$  are the Bernoulli numbers [2].

From (2) it follows

$$\sum_{n=0}^{\infty} (B_n(x+1) - B_n(x)) \frac{t^n}{n!} = te^{tx} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} t^n$$

and we have

$$B_n(x+1) - B_n(x) = nx^{n-1}, \quad n = 0, 1, 2, \dots \quad (4)$$

We use the relations (3) and (4) to conclude that

$$x^n = \sum_{k=0}^n \frac{n! B_k(x)}{k! (n-k+1)!}. \quad (5)$$

## 2. The relation between Laguerre and Bernoulli polynomials

Consider the series

$$\begin{aligned} \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n &= \sum_{n=0}^{\infty} \sum_{s=0}^n \frac{(-1)^s (1+\alpha)_n x^s t^n}{s! (n-s)! (1+\alpha)_s} \\ &= \sum_{n,s=0}^{\infty} \frac{(-1)^s (1+\alpha)_{n+s} x^s t^{n+s}}{s! n! (1+\alpha)_s}. \end{aligned}$$

Using (5) we have

$$\begin{aligned} \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n &= \sum_{n,s=0}^{\infty} \sum_{k=0}^s \frac{(-1)^s (1+\alpha)_{n+s} B_k(x) t^{n+s}}{n! k! (s-k+1)! (1+\alpha)_s} = \\ &= \sum_{n,k,s=0}^{\infty} \frac{(-1)^{s+k} (1+\alpha)_{n+s+k} B_k(x) t^{n+k+s}}{n! k! (s+1)! (1+\alpha)_{s+k}} \end{aligned}$$

in which we use the identity

$$\sum_{n=0}^{\infty} \sum_{k=0}^n A(k, n-k) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) \quad (6)$$

to collect the powers of  $t$  in the last summation above. By the same identity used conversely we may write

$$\begin{aligned} \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n &= \sum_{n,k=0}^{\infty} \sum_{s=0}^n \frac{(-1)^{s+k} (1+\alpha)_{n+k} B_k(x) t^{n+k}}{(n-s)! (1+\alpha)_{s+k} k! (s+1)!} = \\ &= \sum_{n,k=0}^{\infty} {}_2F_2(-n, 1; 1+\alpha+k, 2; 1) \frac{(-1)^k (1+\alpha)_{n+k} B_k(x) t^{n+k}}{n! k! (1+\alpha)_k}. \end{aligned}$$

By the identity (6) used again conversely, we obtain

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^n {}_2F_2(-n+k, 1; 1+\alpha+k, 2; 1) \frac{(-1)^k(1+\alpha)_n B_k(x)t^n}{k!(n-k)!(1+\alpha)_k}.$$

Finally, it is

$$L_n^{(\alpha)}(x) = \sum_{k=0}^n {}_2F_2(-n+k, 1; 1+\alpha+k, 2; 1) \frac{(-1)^k(1+\alpha)_n B_k(x)}{k!(n-k)!(1+\alpha)_k}. \tag{7}$$

### 3. Special cases

1° When  $\alpha = 0$ , we have the simple Laguerre polynomials denoted by  $L_n(x)$  i.e.

$$L_n(x) = L_n^{(0)}(x) = {}_1F_1(-n; 1; x).$$

Then from (7) we obtain

$$L_n(x) = \sum_{k=0}^n {}_2F_2(-n+k, 1; k+1, 2; 1) \frac{(-1)^k n! B_k(x)}{k! k! (n-k)!}.$$

2° Hermite polynomials reduced to Laguerre polynomials [3] give with (7) the relations

$$H_{2n}(x) = \sum_{k=0}^n {}_2F_2(-n+k, 1; k+\frac{1}{2}, 2; 1) \binom{n}{k} \frac{(-1)^{n+k} 2^{n+k} (2n-1)!! B_k(x^2)}{(2k-1)!!}$$

and

$$H_{2n+1}(x) = x \sum_{k=0}^n {}_2F_2(-n+k, 1; k+\frac{3}{2}, 2; 1) \binom{n}{k} \frac{(-1)^{n+k} 2^{n+k+1} (2n+1)!! B_k(x^2)}{(2k+1)!!}.$$

### References

- [1] A. Erdelyi et al.: *Higher Transcendental Functions*, vol. 1, Mc-Graw-Hill, New York-Toronto-London, 1952.
- [2] E.D. Rainville: *Special Functions*, The Macmillan Company, New York, 1960.
- [3] G. Szegő: *Orthogonal Polynomials*, American Mathematical Society, New York, 1950.

## ИЗРАЗУВАЊЕ НА ПОЛИНОМИТЕ НА LAGUERRE СО ПОЛИНОМИТЕ НА BERNOULLI

Благој С. Попов

### Резиме

Во трудот се дава едно развивање на полиномите на Laguerre во редови на полиномите на Bernoulli.

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