A NOTE ABOUT THE SUMS OF BINOMIAL COEFFICIENTS

Билтен на Друштвото на математичарите и физичарите од Н Р Македонија, кн. 4, 1953, 5-6

Here is an alternative, schorter, proof of the result proved by Grosswald[1]).

We have the identity

$$(1)\qquad \left(1-\frac{x}{2}\right)^{2n}\left(1+\frac{x}{1-\frac{x}{2}}\right)^{n}=\left(1-\frac{x^2}{4}\right)^{n}.$$

If we develope here from the binomial theorem, we obtain

$$\sum_{\varkappa=0}\binom{n}{\varkappa}x^{\varkappa}\left(1-\frac{x}{2}\right)^{2n-\varkappa}=\sum_{\nu=0}(-1)^{\nu}\,2^{-2\nu}\binom{n}{\nu}x^{2\nu},$$

i. e.

$$\sum_{\varkappa=0}\binom{n}{\varkappa}x^{\varkappa}\sum_{\lambda=0}(-1)^{\lambda}\,2^{-\lambda}\binom{2n-\varkappa}{\lambda}x^{\lambda}=\sum_{\nu=0}(-1)^{\nu}\,2^{-2\nu}\binom{n}{\nu}x^{2\nu}.$$

Equating the coefficients of x^k, we have

$$\sum_{\lambda=0}(-2)^{-\lambda}\binom{n}{n-\varkappa}\binom{2n-\varkappa}{\lambda}=(-1)^{\nu}\,2^{-2\nu}\binom{n}{\nu},$$

where $\varkappa+\lambda=2\nu=k$.

If we put

$$n-\varkappa=2\nu=\varkappa+\lambda,$$

we have the formula

$$(2)\qquad \sum_{\lambda=0}(-2)^{-\lambda}\binom{n}{\varkappa+\lambda}\binom{n+\varkappa+\lambda}{\lambda}=(-1)^{\nu}\,2^{-2\nu}\binom{n}{\nu},\quad n-\varkappa=2\nu,$$

$$=0\qquad\qquad ,\quad n-\varkappa\neq 2\nu,$$

that Grosswald by means of the Legendre polynomials and the hypergeometric function has proved.

Similarly, we may show that

$$(3)\qquad \sum_{\varkappa=0}(-2)^{\varkappa}\binom{n+\varkappa}{\varkappa}\binom{k+1}{2n+\varkappa+1}=\binom{\frac{k}{2}}{n},\ \text{for } k \text{ even},$$

$$=0\quad,\ \text{for } k \text{ odd}.$$

We perceive, that by a good choose the identity of the form (1), by the showed method, it is possible to get[2]) a great number of formulas of the kind (2) and (3).

REFERENCES

1. E. Grosswald, On sums involving binomial coefficients, The American Mathematical Monthly, vol. 60, p. 179 (1953).

2. E. Netto, Lehrbuch der Combinatorik, B. G. Teubner, Leipzig, 1927.