ON WEBER'S DIFFERENTIAL EQUATION

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THIS paper is concerned with the factorisation of a differential operator. We give the necessary and sufficient condition that a differential operator may be factored into simpler operators.

Consider the generalized Weber's differential equation*

$$L(x, D) \equiv y'' + P_1 y' + P_2 y = 0, \quad D = \frac{d}{dx} \tag{1}$$

where P_i are real polynomials of degree 1, 2 respectively, *i.e.*,

$$P_i(x) = a_{i1}x^2 + a_{i2}x + a_{i3}; \quad a_{11} = 0$$

a_{ik} are arbitrary constants.

If we assume that the operator $L(x, D)$ can be separated into two factors

$$\left(D + f_1(x) + \frac{f_2'(x)}{f_2(x)}\right) \text{ and } \left(D + \frac{f_3(x)}{f_2(x)}\right)$$

then the differential equation (1) may be written in the form

$$L(x, \acute{D})\, y = \left[D + f_1 + \frac{f_2'}{f_2}\right]\left[D + \frac{f_3}{f_2}\right] y = 0 \tag{2}$$

In this case we have the identities {1}

$$\begin{aligned} f_2' + f_3 + f_1 f_2 - P_1 f_2 &\equiv 0 \\ f_3' + f_1 f_3 - P_2 f_2 &\equiv 0 \end{aligned} \tag{3}$$

Let the functions f_i be the polynomials of the form

$$f_i(x) = \sum_{k=0}^{n+1} \beta_{i,k} x^{n-k+1} \tag{4}$$

$$\beta_{1,k} = 0, \ k = 0, 1, \ldots . n - 1, \ \beta_{2,0} = 0, \ \beta_{2,1} = 1$$

Replacing $f_i(x)$ in (3) by its values from (4), we obtain the system of $2n + 5$ non-linear algebraic equations

$$(n - k + 2)\,\beta_{2,k-2} + (\beta_{1,n+1} - a_{13})\,\beta_{2,k-1} + (\beta_{1,n} - a_{22})\,\beta_{2,k} + \beta_{3,k} = 0$$

$$(n - k + 2)\,\beta_{3,k-1} + \beta_{1,n+1}\,\beta_{3,k} + \beta_{1,n}\,\beta_{3,k+1} - a_{21}\,\beta_{2,k+1} - a_{22}\,\beta_{2,k} - a_{23}\,\beta_{2,k-1} = 0,$$

$$(k = 0, 1, 2, \ldots . n + 2).$$

By solving these equations, we get the coefficients of the polynomials

$$\beta_{2,1} = \frac{(a_{12}\,a_{13} - 2a_{22})\,n}{a_{12}^2 - 4a_{21}}, \quad (k + 1)\,\beta_{2,k+1} = \frac{n - k}{k}\,\beta_{2,1}\,\beta_{2,k} \mp \frac{(n - k)(n - k + 1)}{(a_{12}^2 - 4a_{21})^{\frac{1}{2}}}\,\beta_{2,k-1},$$

* The primes denote derivatives with respect to x.

$$2\,\beta_{1,\,n} = a_{12} \mp (a_{12}^2 - 4a_{21})^{\frac{1}{2}},\; 2n\,\beta_{1,\,n+1} = na_{13} \mp (a_{12}^2 - 4a_{21})^{\frac{1}{2}}\,\beta_{2,\,1}\,, \quad (5)$$

$$2\,\beta_{3,\,0} = a_{12} \pm (a_{12}^2 - 4a_{21})^{\frac{1}{2}},$$

$$\beta_{3,\,k} = \beta_{3,\,0}\,\beta_{2,k} + \frac{na_{13} \pm \beta_{2,\,1}(a_{12}^2 - 4a_{21})^{\frac{1}{2}}}{2n}\,\beta_{2,\,k-1} - (n-k+2)\,\beta_{2,\,k-2}\,,$$

$$(k = 1, 2, \ldots n+1)$$

and the relation for the natural number n determining the degree of the polynomials

$$(a_{12}\,a_{13} - 2a_{22})^2 + (a_{12}^2 - 4a_{21})(4a_{23} - 2a_{12} - a_{13}^2)$$
$$= \pm\,2\,(2n+1)\,(a_{12}^2 - 4a_{21})^{3/2}. \quad (6)$$

The preceding relation must be satisfied for the coefficients a_{ik} such that $L(x, D)$ can be separated in the form (2). It is the necessary condition. It is easy to show, with respect to (1), that this condition is sufficient too.

Thus one obtains the following results:

The necessary and sufficient condition that the generalized Weber's equation (1) *may be written in the form* (2) *is that the relation* (6) *is satisfied.*

In this case the functions $f_i\,(x)$ are the polynomials (4), the coefficients of which are determined by (5).

It is obvious that we can solve completely the equation (1), if it can be written in the form (2). According to the relation (6) represents the *integrability condition* too.

For the special Weber's differential equation {2}

$$y'' = \tfrac{1}{4}\,(x^2 + a)\,y,$$

we have from (6) $a = \mp\,2\,(2n+1)$ and we obtain

$$\left[D + \frac{H_n'}{H_n} - \frac{x}{2}\right]\left[D + \frac{R}{H_n}\right] y = 0$$

where

$$R = \tfrac{1}{2}\,x^{n+1} - \binom{n}{1}\frac{n+3}{4}\,x^{n-1} + 3\binom{n}{3}\frac{n+5}{8}\,x^{n-3}\ldots.,$$

$$H_n = He_n\,(x) = (1)^n\,e^{x^2/2}\,D\,(e^{-x^2/2}).$$

Differential equation {3}

$$Y'' + xY' + (n+1)\,Y = 0,$$

may be written in the form

$$\left[D + n\,\frac{H_{n-1}}{H_n}\right]\left[D + \frac{H_{n+1}}{H_n}\right] Y = 0.$$

References

1. Agnev .. *Differential Equations*, 1942, p. 105.
2. Magnus and Oberhettinger *Spezielle Functionen d. math. Phys.*, 1943, p. 91.
3. Kamke .. *Diff.-gleichungen, Lös* III, *Aufl*, 1944, p. 414.