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ANOTHER APPROACH TO COMPLEX INTEGRATION

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Abstract

In this short note we are going to show that the concept of associated differential operators can be used in order to invert complex differential operators by complex line integrals not depending on the path of integration.

While the well–known T_Ω–operator (domain integral over Ω with *Cauchy* singularity, cf. *I. N. Vekua* [4]) defines a special solution of the inhomogeneous *Cauchy–Riemann* equation

$$\partial_{\overline{z}}\, \omega = h \tag{1}$$

if only the right–hand side h is untegrable, a special solution of (1) is also given by the line integral

$$\int_\gamma h(z)\, d\overline{z}$$

provided it does not depend on the path of integration. This is the case if and only if h is anti–holomorphic.

On the other hand, in case h is (locally) a power series in z and $\overline{z}$ (sometimes then one writes $h(z, \overline{z})$ intead of $h(z)$), the inhomogeneous *Cauchy–Riemann* equation (1) can be solved by termwise integration of h. Although not being based on a path–independent line integral, sometimes this solution is denoted by

$$\int_\gamma h(z\,,\overline{z})\, d\overline{z}$$

Let $\mathcal{G}$ be the first order operator

$$\mathcal{G}\omega \equiv \partial_{\overline{z}}\omega - a(z)\omega - b(z)\overline{\omega}.$$

Denote by $\mathcal{W}_{\mathcal{G}}(\Omega)$ the set of all solutions of $\mathcal{G}\omega = 0$ in the domain Ω (note that this differential equation is the canonical form of a uniformly elliptic (homogeneous) system in the plane, see [4]). Then the most general linear and homogeneous first order operator acting on functions $W = W(z)$ belonging to $\mathcal{W}_{\mathcal{G}}(\Omega)$ and not containing $\overline{\partial_z W}$ has the form

$$\mathcal{F}W = C(z)\,\partial_z W + A(z)W + B(z)\,\overline{W}.$$

Notice, further, that the line integral

$$\int_{\gamma} (p\,dz + q\,d\overline{z}) \tag{2}$$

does not depend on γ provided the compatibility condition

$$\partial_{\overline{z}}p - \partial_z q = 0$$

is satisfied (and Ω is simply connected). Consequently, in $\mathcal{W}_{\mathcal{G}}(\Omega)$ the differential operator $\mathcal{F}$ can be inverted by a path–independent line integral of type (2) if the system

$$\mathcal{F}W = \omega \tag{3}$$

$$\mathcal{G}W = 0 \tag{4}$$

is completely integrable where $\omega \in \mathcal{W}_{\mathcal{G}}(\Omega)$ is given. Solving (3), (4) for $\partial_z W$ and $\partial_{\overline{z}}W$, and substituting these expressions into the corresponding integrability condition, one gets four relations for A, B, C, a, b, e.g., C has to be holomorphic and $B = b\overline{C}$.

The question of inverting $\mathcal{F}$ in the space $\mathcal{W}_{\mathcal{G}}(\Omega)$ is closely connected with the construction of associated pairs. A pair $\mathcal{F}, \mathcal{G}$ of first order operators is said to be *associated* if $\mathcal{G} = 0$ implies $\mathcal{G}(\mathcal{F}\omega) = 0$. Since the complex derivative of a holomorphic function is holomorphic again, the operators

$$\mathcal{F} = \partial_z \qquad \text{and} \qquad \mathcal{G} = \partial\overline{z}$$

form an associated pair.

Comparing the conditions for associated pairs formulated in the paper [1] (see also the booklet [2])with the compatibility condition for (3), (4), one gets the following statement:

Theorem. *The compatibility conditions are identical with the conditions for associated pairs.*

This theorem implies, in particular, the following statement:

Provided $\mathcal{F}$, $\mathcal{G}$ is an associated pair, in $\mathcal{W}_{\mathcal{G}}(\Omega)$ the differential operator $\mathcal{F}$ can be inverted by a line integral not depending on the path of integration.

Concluding remarks.

Comparing different inversion formulae (T_Ω–operator, termwise integration of power series, path–independent complex line integrals) and combining them, new versions of complex methods for partial differential equations can be expected.

Representing solutions $W = W(z)$ of (3), (4) in the form (2) (where $W(z)$ can be prescribed at one point z_0), one gets a *Volterra* integral equation which locally can be solved by the contraction–mapping principle provided the equations (3), (4) are compatible (see the paper [3]). The present paper shows that compatibility of (3) an (4) implies that $\mathcal{F}$ and $\mathcal{G}$ are associated, while the converse statement is formulated in [3] already.

Notice, finally, that in the paper [3] also such associated pairs $\mathcal{F}$, $\mathcal{G}$ are under consideration for which $\mathcal{F}$ does contain not only $\partial_z W$ but also $\overline{\partial_z W}$. Moreover, this paper [3] contains analogous constructions for systems in more than two real variables.

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ЕДЕН ПРИОД КОН КОМПЛЕКСНА ИНТЕГРАЦИЈА

В. Тучке

Р е з и м е

Во ова кратка нота се прикажува концептот дека придружниот комплексен диференцијален оператор може да се употреби за пресметување на линиски комплексни интеграли што не зависат од патот на интеграцијата.

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Забелешка на глвниот уредник: Досегашни трудови од оваа област печатени во нашиот Билтен обично даваат неопределен интеграл Векуа, при кој определувањето на аналитичка функција $\phi(z)$ не е тривијален проблем. Во овој труд под некои посебни услови (3) и (4) се дава еден делумен одговор.