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DISTRIBUTIONAL BOUNDARY VALUE OF THE BLASCHKE PRODUCT

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Abstract

In this paper we found the distributional boundary value of the Blaschke product in the upper half plane.

Introduction

We denote by Π^+ the upper half plane, i.e. $\Pi^+ = \{z, \operatorname{Im} z > 0\}$. Suppose that $(\lambda_k)_{k=1}^{\infty}$ is a sequence in the upper half plane such that the series $\sum\limits_{k=1}^{\infty} \operatorname{Im}\lambda_k$ converges. A function of the form

$$b(z) = \prod_{k=1}^{\infty} \frac{z-\lambda_k}{z-\bar{\lambda}_k} \tag{1}$$

is a holomorfic function and is called Blaschke product. Note that $|b(z)| < 1$, for every $z \in \Pi^+$ and

$$\lim_{y \to 0+} b(x+iy) = b^*(x), \quad |b^*(x)| = 1, \ a.e. \tag{2}$$

The functions

$$b_1(z) = \prod_{k=1}^{\infty} \frac{z-\lambda_k}{z-\bar{\lambda}_k} \frac{|1+\lambda_k^2|}{1+\lambda_k^2}, \lambda_k \in \Pi^+, \ k = 1, 2, ..., \ \sum_k \frac{\operatorname{Im}\lambda_k}{|i+\lambda_k|^2} < \infty$$

and

$$b_2(z)=\prod_{k=1}^{\infty}\frac{z-\lambda_k}{z-\bar{\lambda}_k}\frac{\bar{\lambda}_k}{\lambda_k},\ \lambda_k\in\Pi^+,\ k=1,2,...,\ \sum_k\left|\operatorname{Im}\frac{1}{\lambda_k}\right|<\infty$$

are also Blaschke products, $|b_1(z)|<1$, $|b_2(z)|<1$, $z\in\Pi^+$, and satisfy (2). We will work with the Blaschke product (1).

Main result

Let $S(R)$ be the class of rapidly decreasing infinitely differentiable functions, $g\in S(R)$and $f\in H^p(C\backslash R)$. Then the limit

$$f^*(g)=\lim_{y\to 0+}\int_R f(x+iy)\,g(x)\,dx$$

exists and is a distributional boundary value of the function f. The limit

$$T(g)=\lim_{y\to 0+}\int_R [f(x+iy)-f(x-iy)]\,g(x)\,dx$$

exists and is a tempered distribution. Let

$$\lambda_k=\alpha_k+i\beta_k\in\Pi^+,\ k=1,2,....,\ \sum_k\beta_k<\infty.$$

Then, using $\arg 1=0$, we have Djarbashian (see [4]) representation

$$b(z)=\prod_{k=1}^{\infty}\frac{z-\lambda_k}{z-\bar{\lambda}_k}=$$

$$=\exp\left[\sum_{k=1}^{\infty}\int_0^{-\beta_k}\left(\frac{1}{\tau+i(z-\bar{\lambda}_k)}-\frac{1}{\tau+i(z-\lambda_k)}\right)d\tau\right]. \qquad (3)$$

Put

$$l(z,\lambda_k)=\frac{(x-\alpha_k)^2+(y-\beta_k)^2}{(x-\alpha_k)^2+(y+\beta_k)^2},$$

$$p(z,\lambda_k)=\operatorname{arctg}\frac{y-\beta_k}{x-\alpha_k}-\operatorname{arctg}\frac{y+\beta_k}{x-\alpha_k}=$$

$$=-\operatorname{arctg}\frac{2\beta_k(x-\alpha_k)}{(x-\alpha_k)^2+y^2-\beta_k^2}\quad \text{and}$$

$$r(z,\lambda_k)=\log l(z,\lambda_k)+ip(z,\lambda_k).$$

Using the Taylor series develompent of the function $g(x)$ in α_k, we have

$$g(x) = g(\alpha_k) + g'(\alpha_k)(x-\alpha_k) + \frac{g''(\alpha_k)}{2!}(x-\alpha_k)^2 + \cdots =$$
$$= g(\alpha_k) + (x-\alpha_k)\, h(x,\alpha_k),$$

where $h(x,\alpha_k)$ is a bounded function.

Theorem. *Suppose that $b(z)$ is the Blaschke product* (1) *in the upper half plane with zeros $\lambda_k = \alpha_k + i\beta_k$, $k = 1,2,\ldots$, $\sum_{k=1}^{\infty}\beta_k < \infty$ and $g \in S(R)$. Then*

$$b^*(g) = \int_{-\infty}^{\infty} g(x)\,dx - 4\pi\sum_{k=1}^{\infty}\beta_k g(\alpha_k) +$$
$$+ \lim_{y\to 0+}\left\{\sum_{n=2}^{\infty} J(z,\lambda_k,n) + \right.$$
$$+ \lim_{M\to\infty}\sum_{k=1}^{\infty}\int_{-M}^{M}(x-\alpha_k)\log l(z,\lambda_k)\, h(x,\alpha_k)\,dx +$$
$$\left. + i\lim_{M\to\infty}\sum_{k=1}^{\infty}\int_{-M}^{M}(x-a_k)\, h(x,\alpha_k)\, p(z,\lambda_k)\,dx\right\},$$

where

$$J(z,\lambda_k,n) = \int_{-\infty}^{\infty}\frac{1}{n!}\left(\sum_{k=1}^{\infty} r(z,\lambda_k)\right)^n g(x)\,dx,$$

$n \geq 2$ and $p(z,\lambda_k)$, $h(x,\lambda_k)$, $l(z,\lambda_k)$ and $r(z,\lambda_k)$ are defined as above.

Proof. Using the Taylor series develompent of the exponential function and (3), we have

$$b(x+iy) = 1 + \sum_{k=1}^{\infty} r(z,\lambda_k) + \frac{1}{2!}\left(\sum_{k=1}^{\infty} r(z,\lambda_k)\right)^2 + \cdots \tag{4}$$

Using the uniform convergence on the compact subsets, we have

$$I_M(z,\lambda_k)=\int_{-M}^{M}\left(\sum_{k=1}^{\infty}r(z,\lambda_k)\right)g(x)\,dx=$$

$$=\sum_{k=1}^{\infty}\int_{-M}^{M}r(z,\lambda_k)\,g(x)\,dx=$$

$$=\sum_{k=1}^{\infty}\int_{-M}^{M}g(x)\log l(z,\lambda_k)\,dx+i\sum_{k=1}^{\infty}\int_{-M}^{M}g(x)\,p(z,\lambda_k)dx=$$

$$=\sum_{k=1}^{\infty}\int_{-M}^{M}g(\alpha_k)\log l(z,\lambda_k)\,dx+ \tag{5}$$

$$+\sum_{k=1}^{\infty}\int_{-M}^{M}(x-\alpha_k)\,h(x,\alpha_k)\log l(z,\lambda_k)\,dx+$$

$$+i\sum_{k=1}^{\infty}\int_{-M}^{M}g(\alpha_k)p(z,\lambda_k)\,dx+$$

$$+i\sum_{k=1}^{\infty}\int_{-M}^{M}(x-\alpha_k)h(x,\alpha_k)\,p(z,\lambda_k)\,dx\,.$$

For the integrals in (5) we have

$$\int_{-M}^{M}g(\alpha_k)\log l(z,\lambda_k)\,dx=$$

$$=\Bigg[(x-\alpha_k)\log l(z,\lambda_k)+2(y-\beta_k)\operatorname{arctg}\frac{x-\alpha_k}{y-\beta_k}$$

$$-2(y+\beta_k)\operatorname{arctg}\frac{x-\alpha_k}{y+\beta_k}\Bigg]_{-M}^{M}g(\alpha_k)\to$$

$$\to \quad -4\pi\beta_k g(\alpha_k) \text{ as } M\to\infty \qquad \text{and}$$

$$\lim_{M\to+\infty}\int_{-M}^{M}\left(\operatorname{arctg}\frac{y-\beta_k}{x-\alpha_k}-\operatorname{arctg}\frac{y+\beta_k}{x-\alpha_k}\right)dx=0.$$

Hence

$$\lim_{y\to 0+}\lim_{M\to\infty}\int_{-M}^{M}\left(\sum_{k=1}^{\infty}g(\alpha_k)\log l(z,\lambda_k)\,dx\right)=-4\pi\sum_{k=1}^{\infty}\beta_k g(\alpha_k).$$

So using (4), (5) and the above computations, we have

$$\begin{aligned} b^*(g) = &\int_{-\infty}^{\infty} g(x)\,dx - 4\pi\sum_{k=1}^{\infty}\beta_k g(\alpha_k)+ \\ &+\lim_{y\to 0+}\left\{\sum_{n=2}^{\infty}J(z,\lambda_k,n)+\right. \\ &+\lim_{M\to\infty}\sum_{k=1}^{\infty}\int_{-M}^{M}(x-\alpha_k)\log l(z,\lambda_k)\,h(x,\alpha_k)\,dx+ \\ &\left.+i\lim_{M\to\infty}\sum_{k=1}^{\infty}\int_{-M}^{M}(x-a_k)\,h(x,\alpha_k)\,p(z,\lambda_k)\,dx\right\}. \end{aligned}$$

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ГРАНИЧНА ВРЕДНОСТ НА БЛАШКЕОВИОТ ПРОИЗВОД ВО СМИСЛА НА ДИСТРИБУЦИИ

Никола Пандески и Весна Манова-Ераковиќ

Р е з и м е

Во овој труд ја најдовме граничната вредност на Блашкеовиот производ во смисла на дистрибуции.

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