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# WILLMORE SURFACES AND LOOP GROUPS

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## Abstract

After a brief basic review of Willmore surfaces in §1, in §2 is shown that the conformal Gauss map plays a major role in building meromorphic potentials for these surfaces. Section §3 describes the tools of the DPW procedure, which produces the potentials by a loop group technuque. The last section provides concluding remarks which outline essential differences between two perfectible approaches used to develop the presented process.

## 1. Brief account on Willmore surfaces

The study of Willmore surfaces has its origins in the works of G. Thomsen and Shadow [14] in 1923; the existing results were synthetized by W. Blashke in 1929 [2], and reiterated afterwards by K. Voss in 1950 and by T. Willmore in 1960. A recent approach was provided in 1998 by F. Helein [5], based on the works of J. Dorfmeister, F. Pedit and H.-Y. Wu [4], and R. Bryant [3].

Consider a surface $S$ immersed in $\mathbf{R}^3$, oriented and without boundary, and let $\boldsymbol{S}$ be the set of all such surfaces. Let $H$ be the mean curvature, $k_1$ and $k_2$ the principal curvatures, $g$ the genus and $d\sigma$ the area element of $S \in \boldsymbol{S}$. Then, using the Gauss-Bonnet formula, itis straightforward to derive the relation

$$\int_S H^2 d\sigma = \int_S \frac{(k_1 - k_2)^2}{4}\, d\sigma + 4\pi(1-g)\,. \tag{1}$$

where $d\sigma$ is the area element of $S$.

The two integrals above, denoted by $\mathcal{W}_H(S)$ and $\mathcal{W}_k(S)$ respectively, are simultaneously minimized within the family $\boldsymbol{S}$. The first one provides the *Willmore functional*, whose critical points on $S$ are called *Willmore surfaces* [18]. Note that we can also write

$$\mathcal{W}_H(S) = \frac{1}{4}\int_S (k_1^2 + k_2^2)d\sigma + 2\pi(1-g), \qquad (2)$$

hence the right integral has the same critical points as the Willmore functional. By tedious calculation, it can be proved that the Willmore surfaces are the solutions of the 4-th order PDE [18]

$$\Delta H + 2H(H^2 - K) = 0, \qquad (3)$$

where the Laplacian $\Delta$ is constructed using the first fundamental form of $S$.

Hence the *minimal* surfaces are the first (trivial) examples of Willmore surfaces. Still, the class of known examples is wider, since the Willmore functional is invariant under the 10-parameter group of conformal transformations of $\mathbf{R}^3$ ([2] and [14]), whence the conformal transform of any Willmore surface is also a minimizer of $\mathcal{W}_H$.

Moreover, since the stereographic projection is a conformal map, the Willmore surfaces in $\mathbf{R}^3$ are directly related to the ones in $S^3$ or $H^3$. Hence, the stereographic projection to $\mathbf{R}^3$ of compact minimal surfaces in $S^3$ are embedded Willmore surfaces, and it was shown that the area of such a surface $S \subset S^3$ is exactly $\mathcal{W}_H(S)$.

In 1970, H. B. Lawson [10] proved that any compact surface, with the exception of $P^2(\mathbf{R})$, e.g., the 2-holed torus, or the Klein bottle, can be minimally immersed into $S^3$. As consequence, many such compact embedded minimal surfaces in $S^3$ (hence embedded Willmore surfaces) were recently pointed out by H. Karcher, U. Pinkall and I. Sterling [8]. An example of a Willmore surface which *is not* of this type was provided by U. Pinkall [11]. Also, A. Garcia and R. Ruedy proved that any compact two-dimensional Riemannian manifold (like the Wente torus), can be conformally immersed into $\mathbf{R}^3$.

Though trivial, a significant example of Willmore surfaces are *the spheres* in $\mathbf{R}^3$. These are the only totally umbilic Willmore surfaces, and were shown to be the absolute minimizers for $\mathcal{W}_H$ among the surfaces in $S$ of genus 0 [17], a direct computation provides the minimizing value $\mathcal{W}_H(S^2) = 4\pi$ [13].

Also, for $g = 1$, a significant example is the Willmore torus (T. Willmore, 1965), the common torus of revolution in $\mathbf{R}^3$ (of radii $R = \sqrt{2} > r = 1$). An unsolved conjecture (the so-called "Willmore conjecture") states that exactly this torus is the absolute minimizer among the surfaces of genus 1.

The problem of characterizing explicitely the Willmore surfaces of higher genus is still open. Still, complete classification results were obtained for Willmore immersions $r: S^2 \to \mathbf{R}^3$ and $r: P^3(\mathbf{R}) \to \mathbf{R}^3$ by R. Bryant and R. Kustner.

## 2. The conformal Gauss map. Extended frames

Consider a surface $S \in \boldsymbol{S}$, given in conformal coordinates $u = (u^1, u^2) \in D$ by

$$S: r: D \to \mathbf{R}^3, \quad r(u) = \vec{x}(u), \quad \text{for all} \quad u \in D, \tag{4}$$

where $D \ni 0$ is a simply connected domain in $\mathbf{R}^2$. Consider also the inverse of the North stereographic projection $\psi = \Phi^{-1}: \mathbf{R}^3 \to S^3 \subset \mathbf{R}^4$,

$$\psi(\boldsymbol{x}) = \left(\frac{2}{\nu}\boldsymbol{x};\ 1 - \frac{2}{\nu}\right), \quad \text{for all} \quad \boldsymbol{x} \in \mathbf{R}^3, \tag{5}$$

where $\nu = \|\boldsymbol{x}\|^2_{\mathbf{R}^3} + 1$. This mapping identifies the compactification $\mathbf{R}^3 \cup \{\infty\} \underset{\sim}{\overset{\Phi}{\to}} S^3$, and makes possible to embed $S$ into the 5-dimensional Minkowski space $\mathbf{R}^{4,1}$ by the so-called *conformal Gauss map* of $S, \varphi: D \to \mathbf{R}^{4,1}$

$$\varphi(u) = H(u)\big(f(u), 1\big) + \big(n(u), 0\big), \quad \text{for all} \quad u \in D, \tag{6}$$

where $f = \psi \circ r$ and $n: D \to S^3$ is the normal unit vector field of $S$.

Then the hyperquadric $S^{3,1} \cong SO(4,1)/SO(3,1)$ associated with the Lorentz metric

$$\langle(\boldsymbol{x}, x), (\boldsymbol{y}, y)\rangle = \langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\mathbf{R}^4} - xy, \quad \text{for all} \quad ((\boldsymbol{x}, x), (\boldsymbol{y}, y) \in \mathbf{R}^4 \times \mathbf{R} \equiv \mathbf{R}^{4,1} \tag{7}$$

contains the image $\varphi(D)$. Also, the straightforward relation

$$\langle d\varphi, d\varphi\rangle_{\mathbf{R}^{4,1}} = (H^2 - K)d\sigma \tag{8}$$

infers that the energy functional of $\varphi$ is related to $\mathcal{W}_H$ by

$$\int_D \|d\varphi\|^2 d\sigma = \int_D (H^2 - K)d\sigma = \mathcal{W}(S) - 4\pi(1 - g), \tag{9}$$

and hence the Willmore surfaces are characterized by the harmonicity of their conformal Gauss map, being minimizers of the Dirichlet functional of $\varphi$.

For a surface $S \in \boldsymbol{S}$ given by (4), we can consider an associated frame which contains in the first column essentially the position vector of the surface $S$, e.g.,

$$F_0 = \begin{pmatrix} \nu/2 & {}^t\vec{x}F & 1 \\ \vec{x} & F & 0 \\ \frac{\nu}{2} - 1 & {}^t\vec{x}F & 1 \end{pmatrix} \equiv (f_1, f_2, f_3, f_4, f_5) \in \mathcal{G} = QSO(4,1)Q^{-1}, \tag{10}$$

where $Q = \begin{pmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & I_3 & 0 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \end{pmatrix}$, and $F(u) \in SO(3)$ is the orthonormal Gauss frame of the surface $S \in \boldsymbol{S}$ at $u \in D$. The mapping $\tilde{f}_1 : S^3 \subset \mathbf{R}^4 \to \mathbf{R}^{4,1}$,

$$\begin{aligned} &\tilde{f}_1(q) = \\ &= \left(\frac{1+q^4}{\sqrt{2}}, q^1, q^2, q^3, \frac{1-q^4}{\sqrt{2}}\right) \in \mathbf{R}^{4,1}, \text{ for all } q = (q^1, q^2, q^3, q^4) \in S^3 \end{aligned} \tag{11}$$

which induces $f_1$, identifies $S^3$ with the generators of the positive semi-cone of the Lorentz metric (7) and identifies the conformal transformations of $\mathbf{R}^3 \cup \{\infty\} \equiv S^3$ with $QSO(4,1)Q^{-1}$ [5].

Since the conformal Gauss map has values in the reductive homogeneous space

$$S^{3,1} \cong SO(4,1)/SO(3,1),$$

we search for shifted lifts $F = F_0 \cdot g \in QSO(4,1)Q^{-1}$, where $g \in Q\big(SO(2) \times SO(3,1)\big)Q^{-1}$ is a convenient conformal transformation, such that $F$ includes the conformal Gauss map of the surface [5]. Denoting by $\boldsymbol{g}$ and $\boldsymbol{k}$ the Lie algebras of $G = SO(4,1)$ and of the subgroup $K = SO(3,1)$, the Lie algebra of $G$ is subject to the Cartan splitting $\boldsymbol{g} = \boldsymbol{k} \oplus \boldsymbol{p}$,

$$\begin{aligned} \boldsymbol{k} &= \{\xi \in \boldsymbol{g} \mid \mathrm{Ad}_{\mathrm{diag}(I_3,-1,1)}\xi = \xi\} \\ \boldsymbol{p} &= \{\xi \in \boldsymbol{g} \mid \mathrm{Ad}_{\mathrm{diag}(I_3,-1,1)}\xi = -\xi\}. \end{aligned} \tag{12}$$

Then, introducing complex coordinates $(z, \bar{z})$ on $D$, the Maurer-Cartan form $\omega = F^{-1}dF$ associated to the moving frame $F$ splits

$$\omega = \omega'_p + \omega_k + \omega''_p \in \boldsymbol{g} = \boldsymbol{k} \oplus \boldsymbol{p}, \tag{13}$$

where $\omega'_p, \omega''_p \in \boldsymbol{p}$ are the holomorphic/antiholomorphic parts of the $\boldsymbol{p}$-part of $\omega$ and $\omega'_k \in \boldsymbol{k}$.

The idea is then to extend this form by "loopifying" it ([4], [15], [7]), i.e., by introducing the complex parameter $\lambda \in S^1 \subset \mathbf{C}^2$ [4].

$$\omega_\lambda = \lambda^{-1}\omega'_p + \omega_k + \lambda\omega''_p. \tag{14}$$

The conditions under which the equation

$$F_\lambda^{-1} dF_\lambda = \omega_\lambda \tag{15}$$

provides frames $F_\lambda$ associated to Willmore immersions subject to $F_\lambda \mid_{\lambda=1}= F$; $F_\lambda(0) = I_5$, are stated below [5].

**Theorem.** *The first column of* $F_\lambda$ *provides a Willmore immersion iff* $\omega_\lambda$ *has zero curvature, i.e., it satisfies the integrability condition*

$$d\omega_\lambda + \frac{1}{2}[\omega_\lambda \wedge \omega_\lambda] = 0\,. \tag{16}$$

The frames associated to Willmore immersions can be obtained by the DPW loop group procedure, which builds first their corresponding meromorphic potentials and then, after applying a "dressing" method, integrates these potentials to provide new surfaces of the investigated type.

## 3. Loop groups. The DPW procedure

For a given Lie group $G$ considered as subgroup of a matrix group $GL(n, \mathbb{C})$, with the Lie algebra $\boldsymbol{g} = Lie(G) \subset \mathcal{M}_{n\times n}(\mathbb{C})$, we define a structure of Banach space on the family $\Lambda G$ of those mappings

$$g_\lambda\colon S^1 \to G \tag{17}$$

whose natural weighted $H^{1/2}$ and $L^\infty$ norms are finite. These mappings are called *loops* and $\Lambda G$ - a *loop group.* Moreover, $\Lambda G$ has a canonic structure of a (infinite-dimensional) Lie group whose Lie algebra $\Lambda\boldsymbol{g}$ consists of $\boldsymbol{g}$-valued loops of finite norm. The type of loopification described in (14) leads to the consideration of *twisted loop groups.* We define the twisted Lie loop group and its corresponding Lie algebra by [12], [5]

$$\begin{aligned} \Lambda_\sigma G &= \{g \in \Lambda G \mid \sigma g(\lambda)\sigma^{-1} = g(-\lambda)\} \\ \Lambda_\sigma \boldsymbol{g} &= \{\xi \in \Lambda \boldsymbol{g} \mid \sigma\xi(\lambda)\sigma^{-1} = \xi(-\lambda)\}\,, \end{aligned} \tag{18}$$

where $\sigma = diag(I_3, -1, 1)$. Denoting by $G^{\mathbb{C}}$ the complexification of a real Lie group $G$, and by $\boldsymbol{g}^{\mathbb{C}} = \boldsymbol{g} \otimes \mathbb{C}$ its Lie algebra, the Fourier series of a loop $g \in \Lambda_\sigma \boldsymbol{g}^{\mathbb{C}}$ has the even-power coefficients in the subalgebra $\boldsymbol{k}^{\mathbb{C}} \subset \boldsymbol{g}^{\mathbb{C}}$ and the odd ones in the vector space $\boldsymbol{p}^{\mathbb{C}} \subset \boldsymbol{g}^{\mathbb{C}}$.

In the Willmore DPW case we consider

$$\begin{aligned} G &= SO(4,1) \supset K = SO(3,1) \\ G^{\mathbb{C}} &= SO(5,\mathbb{C}) \supset K^{\mathbb{C}} = \big(SO(3,1)\big)^{\mathbb{C}}\,. \end{aligned} \tag{19}$$

The extended frame $F_\lambda$ resulting from the integration of (15) is split *locally* by means of the Birkhoff decomposition

$$F_\lambda = F_\lambda^- \cdot F_\lambda^+ \in \Lambda_\sigma^- G^{\mathbb{C}} \cdot \Lambda_\sigma^+ G^{\mathbb{C}}\,, \tag{20}$$

where

- $\Lambda_\sigma^+ G^{\mathbb{C}} \subset \Lambda_\sigma G^{\mathbb{C}}$ is the subgroup consisting of loops which admit holomorphic extension inside the unit disk and are equal to $I_5$ at the origin, and
- $\Lambda_\sigma^- G^{\mathbb{C}} \subset \Lambda_\sigma G^{\mathbb{C}}$ is the subgroup consisting of loops which admit holomorphic extension outside the unit disk and are equal to $I_5$ at infinity.

Then $\xi = (F_\lambda^-)^{-1} dF_\lambda^-$ is the meromorphic potential associated with the originar map $F$.

The converse procedure emerges with a given meromorphic potential $\xi$. Integrating the equation $\xi = F_-^{-1} dF_-$ provides a complex frame $F_-$, which can be further split using the Iwasawa decomposition [4], [1], [12]

$$F_- = F_* \cdot F_\lambda^+ . \tag{21}$$

Then, the resulting (real) frame $F_*$ includes the conformal Gauss map of a Willmore immersion, which provides the surface.

## 4. Specific features

The choice of the subgroup $K \subset G = SO(4,1)$ in the DPW procedure for Willmore surfaces is not unique. Namely, the first alternative is the one described above, with the subgroup $K = SO(3,1)$ provided by the involution $\sigma$ of $G$. This has the advantage of the density of the "big cell" in $\Lambda G^{\mathbb{C}}$ for the Iwasawa decomposition, but the (major) dissadvantage of untractable decompositions.

Another alternative is the one developed extensively in [5], with $K = SO(3) \times SO(1,1)$ provided by the involution $\tau = diag(-1, I_3, -1)$. Though the decompositions are perfectly feasible, they are just local, and the meromorphic potentials provide finally a frame which differs from a Willmore frame mod a right multiplication with an element of $K$ and the resulting surface has just a so-called "roughly harmonic" conformal Gauss map. Also, in this case, the terms $\omega_p'$ and $\omega_p''$ in the decomposition of the Maurer-Cartan form are nolonger the $(1,0)$- and the $(0,1)$- parts od the component $\omega_p = \omega_p' + \omega_p'' \in \boldsymbol{p}$. Still, the advantage of this method is that the meromorphic potential data provides straightforward the first partials of the immersion, which can be obtained explicitly.

It should be noted that, because of the need to incorporate a *non-degenerate* Gauss map into the moving frame, the Willmore DPW procedure removes from the domian $D$ the umbilic points of the surface $S$, and that the effective results are mainly local.

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# ПОВРШИНИ НА WILLMORE И ЛУПИ

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**Р е з и м е**

По краткиот преглед на површините на Willmore во §1, во §2 се покажува дека конформното Гаусово пресликување игра важна улога во конструкцијата на мероморфните потенцијали за овие површини. Во §3 се опишува суштината на DPW постапката, за создавање потенцијали користејќи лупи. Во последниот параграф се извлечени заклучоци во кои се прикажани суштинските разлики меѓу два совршени пристапа користени во развојот на изложената DPW постапка.

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