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ON GENERALIZATIONS OF DRAGOMIR-AGARWAL INEQUALITY VIA SOME EULER-TYPE IDENTITIES

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Abstract

Some generalizations of Dragomir-Agarwal inequality are given, by using some Euler-type identities.

1. Introduction

One of the cornerstones of nonlinear analysis is the Hadamard inequality, which states that if $[a,b]$, $(a < b)$, is a real interval and $f : [a,b] \to \mathbf{R}$ is a convex function, then

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Recently, S.S. Dragomir and R.P. Agarwal [10] considered the trapezoid formula for numerical integration for functions such that $|f'|^q$ is a convex function for some $q \geq 1$. Their approach was based on estimating the difference between the two sides of the right-hand inequality in (1.1). Improvements of their results were obtained in [13]. In particular, the following tool was established.

Suppose $f : I^0 \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable on I^0 and that $|f'|^q$ is convex on $[a,b]$ for some $q \geq 1$, where $a, b \in I^0 (a < b)$. Then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}. \tag{1.2}$$

Some generalizations of Dragomir-Agarwal inequality for functions which are differentiable with order $n = 2r$, $r \geq 1$ and aplications of these result are given in [4]. See also [5].

In this paper we consider differentiable functions with generally order greater then one, applications of this results and further related results. We will use interval $[0,1]$ because of simplicity and it involves no loss in generality.

2. The main results

In the recent paper [2] the following identities, named the Euler trapezoid formulae, have been proved. If $n = 2r-1$, $r \geq 1$, then

$$\int_0^1 f(t)dt = \frac{1}{2}[f(0)+f(1)] - T_{r-1}(f) + \frac{1}{(2r-1)!}\int_0^1 B_{2r-1}(1-t)\, f^{(2r-1)}(t)dt. \quad (2.1)$$

If $n = 2r$, $r \geq 1$, then

$$\int_0^1 f(t)dt = \frac{1}{2}[f(0)+f(1)] - T_{r-1}(f) + \frac{1}{(2r)!}\int_0^1 [B_{2r}(1-t) - B_{2r}]\, f^{(2r)}dt \quad (2.2)$$

and

$$\int_0^1 f(t)dt = \frac{1}{2}[f(0)+f(1)] - T_r(f) + \frac{1}{(2r)!}\int_0^1 B_{2r}(1-t)\, f^{(2r)}dt, \quad (2.3)$$

where for $0 \leq m \leq \frac{n}{2}$

$$T_m(f) := \sum_{k=1}^{m} \frac{1}{(2k)!} B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right], \quad (2.4)$$

with convention that the sum above is equal to zero when $m = 0$, that is $T_0(f) = 0$.

The identities (2.1), (2.2) and (2.3) hold for every function $f : [0,1] \to \mathbf{R}$ such that $f^{(n)}$ is a continuous function of bounded variation on $[0,1]$, for some $n \geq 1$. Here $B_k(\cdot), k \geq 0$ is the kth Bernoulli polynomial and $B_k = B_k(0) = B_k(1) (k \geq 0)$ the kth Bernoulli number. We denote by $B_k^*(\cdot) (k \geq 0)$ the function of period one with $B_k^*(x) = B_k(x)$ for $0 \leq x \leq 1$.

Theorem 1. *Suppose $f : [0,1] \to \mathbf{R}$ is n-times differentiable.*
(a) *If $\left|f^{(n)}\right|^q$ is convex for some $q \geq 1$, then for $n = 2r-1$, $r \geq 1$, we have*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)] + T_{r-1}\right| \leq \frac{|B_{2r}|}{(2r)!} 4(1-2^{-2r}) \left[\frac{|f^{(2r-1)}(0)|^q + |f^{(2r-1)}(1)|^q}{2}\right]^{\frac{1}{q}}. \quad (2.5)$$

If $n = 2r$, $r \geq 1$, *then*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)]+T_{r-1}\right| \leq \frac{|B_{2r}|}{(2r)!}\left[\frac{|f^{(2r)}(0)|^q+|f^{(2r)}(1)|^q}{2}\right]^{1/q} \tag{2.6}$$

and also, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)]+T_r\right| \leq \frac{2|B_{2r}|}{(2r)!}\left[\frac{|f^{(2r)}(0)|^q+|f^{(2r)}(1)|^q}{2}\right]^{1/q}. \tag{2.7}$$

(b) *If* $|f^{(n)}|^q$ *is concave, then for* $n = 2r-1$, $r \geq 1$, *we have*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)]+T_{r-1}\right| \leq \frac{|B_{2r}|}{(2r)!}4(1-2^{-2r})\left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|. \tag{2.8}$$

If $n = 2r$, $r \geq 1$, *then*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)]+T_{r-1}\right| \leq \frac{|B_{2r}|}{(2r)!}\left|f^{(2r)}\left(\frac{1}{2}\right)\right| \tag{2.9}$$

and also, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)]+T_r\right| \leq \frac{2|B_{2r}|}{(2r)!}\left|f^{(2r)}\left(\frac{1}{2}\right)\right|. \tag{2.10}$$

Proof. Let H_r be defined as

$$\begin{aligned} H_r &= (-1)^r\left\{\int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)]+T_{r-1}\right\} \\ &= (-1)^r\frac{1}{(2r-1)!}\int_0^1 B_{2r-1}(1-t)f^{(2r-1)}(t)dt. \end{aligned} \tag{2.11}$$

Because of [11, 23.1.14,23.1.8]

$$(-1)^r B_{2r-1}(t) > 0,\ 0 < t < \frac{1}{2},\ r \geq 2$$

and

$$B_{2r-1}(1-t) = -B_{2r-1}(t),\ t \in [0,1].$$

In [2] has been calculated that

$$\int_0^1 |B_{2r-1}(t)|dt = \frac{2(1-2^{-2r})}{r}| B_{2r}|.$$

Note that

$$\int_0^1 t|B_{2r-1}(t)|dt + \int_0^1 (1-t)|B_{2r-1}(t)|dt = \int_0^1 |B_{2r-1}(t)|dt = \frac{2(1-2^{-2r})}{r}|B_{2r}|. \tag{2.12}$$

On the other hand, by partial integration we have

$$\begin{aligned} \int_0^1 t|B_{2r-1}(t)|dt &= \int_0^{1/2} t|B_{2r-1}(t)|dt + \int_{1/2}^1 t|B_{2r-1}(t)|dt \\ &= \left|\int_0^{1/2} tB_{2r-1}(t)dt\right| + \left|\int_{1/2}^1 tB_{2r-1}(t)dt\right| \\ &= \left|\frac{1}{2r}tB_{2r}(t)|_0^{1/2} - \frac{1}{2r}\int_0^{1/2} B_{2r}(t)dt\right| \\ &+ \left|\frac{1}{2r}tB_{2r}(t)|_{1/2}^1 - \frac{1}{2r}\int_{1/2}^1 B_{2r}(t)dt\right| \qquad (2.13) \\ &= \frac{1}{4r}\left|B_{2r}\left(\frac{1}{2}\right)\right| + \frac{1}{2r}\left|B_{2r} - \frac{1}{2}B_{2r}\left(\frac{1}{2}\right)\right| \\ &= \frac{1}{4r}\left|-(1-2^{1-2r})B_{2r}\right| + \frac{1}{2r}\left|B_{2r} + \frac{1}{2}(1-2^{1-2r})B_{2r}\right| \\ &= \frac{1-2^{-2r}}{r}|B_{2r}|, \end{aligned}$$

where we have used substitutions $u = t$, $dv = B_{2r-1}(t)dt$ so that $du = dt$ and $v = B_{2r}(t)/2r$.

With respect of (2.12), we have

$$\int_0^1 t|B_{2r-1}(t)|dt = \int_0^1 (1-t)|B_{2r-1}(t)|dt = \frac{1}{2}\int_0^1 |B_{2r-1}(t)|dt.$$

So, by power mean inequality we have

$$\begin{aligned} |H_r| &\le \frac{1}{(2r-1)!}\int_0^1 |B_{2r-1}(t)| \cdot \left|f^{(2r-1)}(t)\right| dt \\ &\le \left(\frac{1}{(2r-1)!}\int_0^1 |B_{2r-1}(t)|\, dt\right)^{1-1/q} \left(\frac{1}{(2r-1)!}\int_0^1 |B_{2r-1}(t)| \cdot |f^{(2r-1)}(t)|^q dt\right)^{\frac{1}{q}}. \end{aligned}$$

Now, by Jensen's inequality

$$
\begin{aligned}
|H_r| &\le \left(\frac{1}{(2r-1)!}\int_0^1 |B_{2r-1}(t)|\,dt\right)^{1-1/q} \times \\
&\times \left(\frac{1}{(2r-1)!}\int_0^1 |B_{2r-1}(t)|\cdot\left[t\left|f^{(2r-1)}(1)\right|^q + (1-t)\left|f^{(2r-1)}(0)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{(2r-1)!}\int_0^1 |B_{2r-1}(t)|\,dt\right)^{1-1/q} \times \\
&\times \left(\frac{1}{(2r-1)!}\left[\left|f^{(2r-1)}(1)\right|^q \int_0^1 t|B_{2r-1}(t)|dt\right.\right. \\
&\left.\left.+ \left|f^{(2r-1)}(0)\right|^q \int_0^1 (1-t)|B_{2r-1}(t)|dt\right]\right)^{\frac{1}{q}} \qquad (2.14) \\
&= \left(\frac{|B_{2r}|}{(2r)!}4(1-2^{-2r})\right)^{1-1/q} \times \\
&\times \left(\frac{|B_{2r}|}{(2r)!}2(1-2^{-2r})\left|f^{(2r-1)}(1)\right|^q + \frac{|B_{2r}|}{(2r)!}2(1-2^{-2r})\left|f^{(2r-1)}(0)\right|^q\right)^{\frac{1}{q}} \\
&= \frac{|B_{2r}|}{(2r)!}4(1-2^{-2r})\left[\frac{\left|f^{(2r-1)}(0)\right|^q + \left|f^{(2r-1)}(1)\right|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}
$$

Moreover, if $\left|f^{(2r-1)}\right|^q$ is concave then

$$
\begin{aligned}
|H_r| &\le \frac{1}{(2r-1)!}\int_0^1 |B_{2r-1}(t)|\cdot\left|f^{(2r-1)}(t)\right|dt \le \\
&\le \left(\frac{1}{(2r-1)!}\int_0^1 |B_{2r-1}(t)|\,dt\right)\left[\left|f^{(2r-1)}\left(\frac{\int_0^1 |B_{2r-1}(t)|\,tdt}{\int_0^1 |B_{2r-1}(t)|\,dt}\right)\right|^q\right]^{1/q} \\
&= \frac{|B_{2r}|}{(2r)!}4(1-2^{-2r})\left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|
\end{aligned}
$$

and the inequalities (2.5) and (2.8) are completely proved.

Proofs of the inequalities (2.6) and (2.9) are similarly. See also [4].

To proof (2.7) and (2.10) we use the same arguments with

$\int_0^1 |B_{2r}(t)|dt \le 2|B_{2r}|$. □

Remark 1. *For (2.8), (2.9) and (2.10) to be satisfied it is enough to suppose that* $\left|f^{(n)}\right|$ *is a concave function. In fact, if* $|g|^q$ *is concave on* $[0,1]$, *for some* $q \geq 1$, *then for* $x, y, \lambda \in [0,1]$

$$\begin{aligned} |g(\lambda x + (1-\lambda)y)|^q &\geq \lambda |g(x)|^q + (1-\lambda)|g(y)|^q \\ &\geq (\lambda |g(x)| + (1-\lambda)|g(y)|)^q \end{aligned}$$

by the power mean inequality. Therefore, $|g|$ *is also concave on* $[0,1]$.

Remark 2. *It is simply to prove that if* $f : [a,b] \to \mathbf{R}$ *is* n*-times differentiable then*
(a) *If* $\left|f^{(n)}\right|^q$ *is convex for some* $q \geq 1$, *then for* $n = 2r-1$, $r \geq 1$, *we have*

$$|H_r(a,b)| \leq (b-a)^{2r} \frac{|B_{2r}|}{(2r)!} 4(1-2^{-2r}) \left[\frac{\left|f^{(2r-1)}(a)\right|^q + \left|f^{(2r-1)}(b)\right|^q}{2}\right]^{1/q}.$$

If $n = 2r$, $r \geq 1$, *then*

$$|H_r(a,b)| \leq (b-a)^{2r+1} \frac{|B_{2r}|}{(2r)!} \left[\frac{\left|f^{(2r)}(a)\right|^q + \left|f^{(2r)}(b)\right|^q}{2}\right]^{1/q}$$

and also, we have

$$|H_r^*(a,b)| \leq (b-a)^{2r+1} \frac{2|B_{2r}|}{(2r)!} \left[\frac{\left|f^{(2r)}(a)\right|^q + \left|f^{(2r)}(b)\right|^q}{2}\right]^{1/q}.$$

(b) *If* $\left|f^{(n)}\right|$ *is concave, then for* $n = 2r-1$, $r \geq 1$, *we have*

$$|H_r(a,b)| \leq (b-a)^{2r} \frac{|B_{2r}|}{(2r)!} 4(1-2^{-2r}) \left|f^{(2r-1)}\left(\frac{a+b}{2}\right)\right|.$$

If $n = 2r$, $r \geq 1$, *then*

$$|H_r(a,b)| \leq (b-a)^{2r+1} \frac{|B_{2r}|}{(2r)!} \left|f^{(2r)}\left(\frac{a+b}{2}\right)\right|$$

and also, we have

$$|H_r^*(a,b)| \leq (b-a)^{2r+1} \frac{2|B_{2r}|}{(2r)!} \left|f^{(2r)}\left(\frac{a+b}{2}\right)\right|,$$

where

$$H_r(a,b) := (-1)^r \left\{ \int_a^b f(t)dt - \frac{b-a}{2}[f(a)+f(b)] + \right.$$
$$\left. + \sum_{k=1}^{r-1} \frac{B_{2k}(b-a)^{2k}}{(2k)!} \left[f^{(2k-1)}(b) - f^{(2k-1)}(a) \right] \right\}$$

and

$$H_r^*(a,b) := (-1)^r \left\{ \int_a^b f(t)dt - \frac{b-a}{2}[f(a)+f(b)] + \right.$$
$$\left. + \sum_{k=1}^{r} \frac{B_{2k}(b-a)^{2k}}{(2k)!} \left[f^{(2k-1)}(b) - f^{(2k-1)}(a) \right] \right\}.$$

The resultant formulae in Theorem 1 when $n = 1$ are of the special interest and we isolate them as corollary.

Corollary 1. *Suppose $f : [0,1] \to \mathbf{R}$ is differentiable.*
(a) *If $|f'|^q$ is convex for some $q \geq 1$, then*

$$\left| \int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)] \right| \leq \frac{1}{4} \left[\frac{|f'(0)|^q + |f'(1)|^q}{2} \right]^{1/q}.$$

(b) *If $|f'|$ is concave, then*

$$\left| \int_0^1 f(t)dt - \frac{1}{2}[f(0)+f(1)] \right| \leq \frac{1}{4} \left| f'\left(\frac{1}{2}\right) \right|.$$

Remark 3. *The inequalities in Theorem 1 are generalizations of the Dragomir-Agarwal inequality, because for $n = 1$ (see (a) in above corollary) we get* (1.2).

3. The Euler midpoint formulae

In this section we take a different path from (2.1), (2.2) and (2.3), one leading to the Euler midpoint formulae instead of the Euler trapezoid formulae.

In the recent paper [3] the following identity, named the Euler midpoint formulae, have been proved. If $n = 2r-1$, $r \geq 1$, then

$$\int_0^1 f(t)dt = f\left(\frac{1}{2}\right) - T_{r-1}^M(f) + \frac{1}{(2r-1)!}\int_0^1 B_{2r-1}^*\left(\frac{1}{2}-t\right) f^{(2r-1)}(t)dt.$$

If $n = 2r$, $r \geq 1$, then

$$\int_0^1 f(t)dt = f\left(\frac{1}{2}\right) - T_{r-1}^M(f) + \frac{1}{(2r)!}\int_0^1 \left[B_{2r}^*\left(\frac{1}{2}-t\right) - B_{2r}\left(\frac{1}{2}\right)\right] f^{(2r)}dt,$$

and

$$\int_0^1 f(t)dt = f\left(\frac{1}{2}\right) - T_r^M(f) + \frac{1}{(2r)!}\int_0^1 B_{2r}^*\left(\frac{1}{2}-t\right) f^{(2r)}dt,$$

where for $0 \leq m \leq \frac{n}{2}$

$$T_m^M(f) := \sum_{k=1}^{m} \frac{1}{(2k)!} B_{2k}(1-2^{1-2k})\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right],$$

with convention that the sum above is equal to zero when $m = 0$, that is $T_0^M(f) = 0$.

In [3] have been calculated that

$$\int_0^1 \left|B_{2r-1}^*\left(\frac{1}{2}-t\right)\right| dt = \frac{2(1-2^{-2r})}{r}|B_{2r}|,$$

$$\int_0^1 \left|B_{2r}^*\left(\frac{1}{2}-t\right) - B_{2r}\left(\frac{1}{2}\right)\right| dt = (1-2^{1-2r})|B_{2r}|, \quad \text{see also [5]}$$

and

$$\int_0^1 \left|B_{2r}^*\left(\frac{1}{2}-t\right)\right| dt \leq 2|B_{2r}|.$$

We can parallel the development of the previous section with the following theorem.

Theorem 2. *Suppose $f : [0,1] \to \mathbf{R}$ is n-times differentiable.*
(a) *If $\left|f^{(n)}\right|^q$ is convex for some $q \geq 1$, then for $n = 2r-1$, $r \geq 1$, we have*

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right) + T_{r-1}^M\right| \leq \frac{|B_{2r}|}{(2r)!}4(1-2^{-2r})\left[\frac{|f^{(2r-1)}(0)|^q + |f^{(2r-1)}(1)|^q}{2}\right]^{\frac{1}{q}}.$$

If $n = 2r$, $r \geq 1$, then

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right) + T_{r-1}^M\right| \leq \frac{|B_{2r}|}{(2r)!}(1-2^{1-2r})\left[\frac{\left|f^{(2r)}(0)\right|^q + \left|f^{(2r)}(1)\right|^q}{2}\right]^{\frac{1}{q}}$$

and also, we have

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right) + T_r^M\right| \leq \frac{2|B_{2r}|}{(2r)!}\left[\frac{|f^{(2r)}(0)|^q + |f^{(2r)}(1)|^q}{2}\right]^{1/q}.$$

(b) *If* $|f^{(n)}|^q$ *is concave, then for* $n = 2r-1,\ r \geq 1$, *we have*

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right) + T_{r-1}^M\right| \leq \frac{|B_{2r}|}{(2r)!}4(1-2^{-2r})\left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|,$$

If $n = 2r,\ r \geq 1$, *then*

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right) + T_{r-1}^M\right| \leq \frac{|B_{2r}|}{(2r)!}(1-2^{1-2r})\left|f^{(2r)}\left(\frac{1}{2}\right)\right|$$

and also, we have

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right) + T_r^M\right| \leq \frac{2|B_{2r}|}{(2r)!}\left|f^{(2r)}\left(\frac{1}{2}\right)\right|.$$

The resultant formulae in Theorem 2 when $n = 1$ *are of the special interest and we isolate them as corollary.*

Corollary 2. *Suppose* $f : [0,1] \to \mathbf{R}$ *is differentiable.*
(a) *If* $|f'|^q$ *is convex for some* $q \geq 1$, *then*

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right)\right| \leq \frac{1}{4}\left[\frac{|f'(0)|^q + |f'(1)|^q}{2}\right]^{1/q}.$$

(b) *If* $|f'|$ *is concave, then*

$$\left|\int_0^1 f(t)dt - f\left(\frac{1}{2}\right)\right| \leq \frac{1}{4}\left|f'\left(\frac{1}{2}\right)\right|.$$

4. The Euler-Simpson formulae

In this section we explore a path that is associated with the Euler-Simpson formulae.

If f is defined on segment $[0,1]$ and has n ($n \geq 3$) continuous derivatives there, then the Euler-Simpson formulae (see [6]) state that if $n = 2r-1,\ r \geq 2$

$$\int_0^1 f(t)dt = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] + T_r^X(f) + \frac{1}{3(2r-1)!}\int_0^1 f^{(2r-1)}(t)F_{2r-1}(t)dt.$$

If $n = 2r,\ r \geq 2$, then

$$\int_0^1 f(t)dt = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] + T_r^X(f) + \frac{1}{3(2r)!}\int_0^1 f^{(2r)}(t)F_{2r}(t)dt$$

and

$$\int_0^1 f(t)dt = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] + T_{r+1}^X(f) + \frac{1}{3(2r)!}\int_0^1 f^{(2r)}(t)G_{2r}(t)dt,$$

where $T_0^X(f) = T_1^X(f) = 0$ and for $1 \leq m \leq \frac{n}{2}$

$$T_m^X(f) = \sum_{k=1}^{m-1} \frac{1}{3(2k)!}(1 - 2^{2-2k})B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]$$

and

$$G_n(t) = B_n^*(1-t) + 2B_n^*\left(\frac{1}{2} - t\right),$$

$$F_n(t) = B_n^*(1-t) + 2B_n^*\left(\frac{1}{2} - t\right) - B_n - 2B_n\left(\frac{1}{2}\right).$$

It was proved in [11] that $F_n(1-t) = (-1)^n F_n(t)$ and that $(-1)^{r-1}F_{2r}(t) \geq 0$. Also (see [6]),

$$\int_0^1 |F_{2r-1}(t)|\,dt = \frac{2 - 2^{1-2r}}{r}|B_{2r}|,$$

$$\int_0^1 |F_{2r}(t)|\,dt = (1 - 2^{2-2r})|B_{2r}|$$

and

$$\int_0^1 |G_{2r}(t)|\,dt \leq 2(1 - 2^{2-2r})|B_{2r}|.$$

We can parallel the development of the second section with the following theorem.

Theorem 3. *Suppose $f : [0,1] \to \mathbf{R}$ is n-times differentiable.*
(a) *If $\left|f^{(n)}\right|^q$ is convex for some $q \geq 1$, then for $n = 2r - 1,\ r \geq 2$, we have*

$$\left|\int_0^1 f(t)dt - \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - T_r^X(f)\right|$$
$$\leq \frac{|B_{2r}|}{3(2r)!}2(2 - 2^{1-2r})\left[\frac{\left|f^{(2r-1)}(0)\right|^q + \left|f^{(2r-1)}(1)\right|^q}{2}\right]^{1/q}.$$

If $n = 2r,\ r \geq 2,$ *then*

$$\left|\int_0^1 f(t)dt - \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - T_r^X(f)\right|$$
$$\leq \frac{|B_{2r}|}{3(2r)!}(1 - 2^{2-2r})\left[\frac{\left|f^{(2r)}(0)\right|^q + \left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q}$$

and also, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - T_{r+1}^X(f)\right|$$
$$\leq \frac{|B_{2r}|}{3(2r)!}2(1 - 2^{2-2r})\left[\frac{\left|f^{(2r)}(0)\right|^q + \left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q}.$$

(b) *If* $\left|f^{(n)}\right|$ *is concave, then for* $n = 2r - 1,\ r \geq 2,$ *we have*

$$\left|\int_0^1 f(t)dt - \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - T_r^X(f)\right|$$
$$\leq \frac{|B_{2r}|}{3(2r)!}2(2 - 2^{1-2r})\left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|.$$

If $n = 2r, r \geq 2,$ *then*

$$\left|\int_0^1 f(t)dt - \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - T_r^X(f)\right|$$
$$\leq \frac{|B_{2r}|}{3(2r)!}(1 - 2^{2-2r})\left|f^{(2r)}\left(\frac{1}{2}\right)\right|$$

and also, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] - T_{r+1}^X(f)\right|$$
$$\leq \frac{|B_{2r}|}{3(2r)!}2(1 - 2^{2-2r})\left|f^{(2r)}\left(\frac{1}{2}\right)\right|.$$

The resultant formulae in Theorem 3 when $n = 3$ are of special interest and we isolate them as corollary.

Corollary 3. *Suppose* $f : [0, 1] \to \mathbf{R}$ *is 3-times differentiable.*

(a) *If* $\left|f^{(3)}\right|^q$ *is convex for some* $q \geq 1$, *then*

$$\left|\int_0^1 f(t)dt - \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]\right| \leq \frac{1}{576}\left[\frac{|f^{(3)}(0)|^q + |f^{(3)}(1)|^q}{2}\right]^{\frac{1}{q}}.$$

(b) *If* $\left|f^{(3)}\right|$ *is concave, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]\right| \leq \frac{1}{576}\left|f^{(3)}\left(\frac{1}{2}\right)\right|.$$

5. The Euler two-point formulae

In this section we explore a path that is associated with the Euler two-point formulae.

If f is defined on segment $[0,1]$ and has n ($n \geq 3$) continuous derivatives there, then the Euler two-point formulae (see [12]) state that if $n = 2r-1$, $r \geq 2$

$$\int_0^1 f(t)dt = \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right] + T_{r-1}^P(f) + \frac{1}{2(2r-1)!}\int_0^1 f^{(2r-1)}(t)F_{2r-1}^P(t)dt.$$

If $n = 2r$, $r \geq 2$, then

$$\int_0^1 f(t)dt = \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right] + T_{r-1}^P(f) + \frac{1}{2(2r)!}\int_0^1 f^{(2r)}(t)F_{2r}^P(t)dt$$

and

$$\int_0^1 f(t)dt = \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right] + T_r^P(f) + \frac{1}{2(2r)!}\int_0^1 f^{(2r)}(t)G_{2r}^P(t)dt,$$

where $T_0^P(f) = 0$ and for $1 \leq m \leq \frac{n}{2}$

$$T_m^P(f) = \sum_{k=1}^m \frac{1}{2(2k)!}(1 - 3^{1-2k})B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]$$

and

$$G_n^P(t) = B_n^*\left(\frac{1}{3} - t\right) + B_n^*\left(\frac{2}{3} - t\right),$$

$$F_n^P = B_n^*\left(\frac{1}{3} - t\right) + B_n^*\left(\frac{2}{3} - t\right) - B_n\left(\frac{1}{3}\right) - B_n\left(\frac{2}{3}\right).$$

It was proved in [12] that $F_n^P(1-t)=(-1)^n F_n^P(t)$ and that $(-1)^r F_{2r-1}(t)\geq 0$ in $\left[0,\frac{1}{2}\right]$. Also

$$\int_0^1 \left|F_{2r-1}^P(t)\right| dt = \frac{2}{r}(1-2^{-2r})(1-3^{1-2r})|B_{2r}|,$$

$$\int_0^1 \left|F_{2r}^P(t)\right| dt = (1-3^{1-2r})|B_{2r}|$$

and

$$\int_0^1 \left|F_{2r}^P(t)\right| dt \leq 2(1-3^{1-2r})|B_{2r}|.$$

We can parallel the development of the second section with the following theorem.

Theorem 4. *Suppose $f:[0,1]\to\mathbf{R}$ is n-times differentiable.*
(a) *If $\left|f^{(n)}\right|^q$ is convex for some $q\geq 1$, then for $n=2r-1$, $r\geq 2$, we have*

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f\left(\frac{1}{3}\right)+f\left(\frac{2}{3}\right)\right]-T_{r-1}^P(f)\right|$$
$$\leq\frac{|B_{2r}|}{(2r)!}2(1-2^{-2r})(1-3^{1-2r})\left[\frac{\left|f^{(2r-1)}(0)\right|^q+\left|f^{(2r-1)}(1)\right|^q}{2}\right]^{1/q}.$$

If $n=2r$, $r\geq 2$, then

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f\left(\frac{1}{3}\right)+f\left(\frac{2}{3}\right)\right]-T_{r-1}^P(f)\right|$$
$$\leq\frac{|B_{2r}|}{2(2r)!}(1-3^{1-2r})\left[\frac{\left|f^{(2r)}(0)\right|^q+\left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q}$$

and also, we have

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f\left(\frac{1}{3}\right)+f\left(\frac{2}{3}\right)\right]-T_r^P(f)\right|$$
$$\leq\frac{|B_{2r}|}{(2r)!}(1-3^{1-2r})\left[\frac{\left|f^{(2r)}(0)\right|^q+\left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q}.$$

(b) *If $\left|f^{(2r)}\right|$ is concave, then*

$$\left|\int_0^1 f(t)dt-\frac{1}{2}\left[f\left(\frac{1}{3}\right)+f\left(\frac{2}{3}\right)\right]-T_{r-1}^P(f)\right|$$
$$\leq\frac{|B_{2r}|}{(2r)!}2(1-2^{-2r})(1-3^{1-2r})\left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|.$$

If $n = 2r$, $r \geq 2$, *then*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right] - T_{r-1}^P(f)\right| \leq \frac{|B_{2r}|}{2(2r)!}(1 - 3^{1-2r})\left|f^{(2r)}\left(\frac{1}{2}\right)\right|$$

and also, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right] - T_r^P(f)\right| \leq \frac{|B_{2r}|}{(2r)!}(1 - 3^{1-2r})\left|f^{(2r)}\left(\frac{1}{2}\right)\right|.$$

The resultant formulae in Theorem 4 when $n = 3$ are of special interest and we isolate them as corollary.

Corollary 4. *Suppose* $f : [0, 1] \to \mathbf{R}$ *is 3-times differentiable.*
(a) *If* $\left|f^{(3)}\right|^q$ *is convex for some* $q \geq 1$, *then*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right]\right| \leq \frac{13}{5184}\left[\frac{\left|f^{(3)}(0)\right|^q + \left|f^{(3)}(1)\right|^q}{2}\right]^{1/q}.$$

(b) *If* $\left|f^{(3)}\right|$ *is concave, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{2}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right)\right]\right| \leq \frac{13}{5184}\left|f^{(3)}\left(\frac{1}{2}\right)\right|.$$

6. The dual Euler-Simpson formulae

In this section we explore a path that is associated with the dual Euler-Simpson formulae.

If f is defined on segment $[0, 1]$ and has n ($n \geq 3$) continuous derivatives there, then the dual Euler-Simpson formulae (see [7]) state that if $n = 2r - 1$, $r \geq 2$

$$\int_0^1 f(t)dt = \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] - T_{r-1}^D(f) + \frac{1}{3(2r-1)!}\int_0^1 f^{(2r-1)}(t)F_{2r-1}^D(t)dt.$$

If $n = 2r$, $r \geq 2$, then

$$\int_0^1 f(t)dt = \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] - T_{r-1}^D(f) + \frac{1}{3(2r)!}\int_0^1 f^{(2r)}(t)F_{2r}^D(t)dt$$

and

$$\int_0^1 f(t)dt = \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] - T_r^D(f) + \frac{1}{3(2r)!}\int_0^1 f^{(2r)}(t)G_{2r}^D(t)dt.$$

where $T_0^D(f) = 0$ and for $1 \leq m \leq \frac{n}{2}$

$$T_m^D(f) = \sum_{k=1}^{m} \frac{1}{3(2k)!}(2^{3-4k} - 3 \cdot 2^{1-2k} + 1)B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]$$

and

$$G_n^D = 2B_n^*\left(\frac{1}{4} - t\right) - B_n^*\left(\frac{1}{2} - t\right) + 2B_n^*\left(\frac{3}{4} - t\right),$$

$$F_n^D = 2B_n^*\left(\frac{1}{4} - t\right) - B_n^*\left(\frac{1}{2} - t\right) + 2B_n^*\left(\frac{3}{4} - t\right) - 2B_n\left(\frac{1}{4}\right) + B_n\left(\frac{1}{2}\right) - 2B_n\left(\frac{3}{4}\right).$$

It was proved in [7] that $F_n^D(1-t)=(-1)^n F_n^D(t)$ and that $(-1)^{r-1}F_{2r-1}^D(t)\geq 0$ in $\left[0, \frac{1}{2}\right]$. Also

$$\int_0^1 \left|F_{2r-1}^D(t)\right| dt = \frac{2}{r}(1 - 2^{-2r})|B_{2r}|,$$

$$\int_0^1 \left|F_{2r}^D(t)\right| dt = (2^{3-4r} - 3 \cdot 2^{1-2r} + 1)|B_{2r}|$$

and

$$\int_0^1 \left|G_{2r}^D(t)\right| dt \leq 2(2^{3-4r} - 3 \cdot 2^{1-2r} + 1)|B_{2r}|.$$

We can parallel the development of the second section with the following theorem.

Theorem 5. *Suppose $f : [0,1] \to \mathbf{R}$ is n-times differentiable.*
(a) If $\left|f^{(n)}\right|^q$ is convex for some $q \geq 1$, then if $n = 2r - 1,\ r \geq 2$, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] + T_{r-1}^D(f)\right| \leq \frac{|B_{2r}|}{3(2r)!}4(1 - 2^{-2r})\left[\frac{\left|f^{(2r-1)}(0)\right|^q + \left|f^{(2r-1)}(1)\right|^q}{2}\right]^{1/q}.$$

If $n = 2r,\ r \geq 2$, *then*

$$\left|\int_0^1 f(t)dt - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] + T_{r-1}^D(f)\right|$$
$$\leq \frac{|B_{2r}|}{3(2r)!}(2^{3-4r} - 3 \cdot 2^{1-2r} + 1)\left[\frac{\left|f^{(2r)}(0)\right|^q + \left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q}$$

and also, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] + T_r^D(f)\right|$$
$$\leq \frac{|B_{2r}|}{3(2r)!}2(2^{3-4r} - 3 \cdot 2^{1-2r} + 1)\left[\frac{\left|f^{(2r)}(0)\right|^q + \left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q}.$$

(b) *If* $\left|f^{(n)}\right|$ *is concave, then for* $n = 2r - 1,\ r \geq 2$, *we have*

$$\left|\int_0^1 f(t)dt - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] + T_{r-1}^D(f)\right|$$
$$\leq \frac{|B_{2r}|}{3(2r)!}4(1 - 2^{-2r})\left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|.$$

If $n = 2r,\ r \geq 2$, *then*

$$\left|\int_0^1 f(t)dt - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] + T_{r-1}^D(f)\right|$$
$$\leq \frac{|B_{2r}|}{3(2r)!}(2^{3-4r} - 3 \cdot 2^{1-2r} + 1)\left|f^{(2r)}\left(\frac{1}{2}\right)\right|$$

and also, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] + T_{r-1}^D(f)\right|$$
$$\leq \frac{|B_{2r}|}{3(2r)!}2(2^{3-4r} - 3 \cdot 2^{1-2r} + 1)\left|f^{(2r)}\left(\frac{1}{2}\right)\right|.$$

The resultant formulae in Theorem 5 when $n = 3$ are of special interest and we isolate them as corollary.

Corollary 5. *Suppose* $f : [0, 1] \to \mathbf{R}$ *is* 3*-times differentiable.*

(a) *If* $\left|f^{(3)}\right|^q$ *is convex for some* $q \geq 1$, *then*

$$\left|\int_0^1 f(t)dt - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right]\right| \leq \frac{1}{576}\left[\frac{|f^{(3)}(0)|^q + |f^{(3)}(1)|^q}{2}\right]^{\frac{1}{q}}.$$

(b) *If* $\left|f^{(3)}\right|$ *is concave, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right]\right| \leq \frac{1}{576}\left|f^{(3)}\left(\frac{1}{2}\right)\right|.$$

7. The Euler-Simpson 3/8 formulae

In this section we explore a path that is associated with the Euler-Simpson 3/8 formulae.

If f is defined on segment $[0,1]$ and has n ($n \geq 3$) continuous derivatives there, then the Euler-Simpson 3/8 formulae (see [8]) state that if $n = 2r-1$, $r \geq 2$

$$\int_0^1 f(t)dt = \frac{1}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right] + T_{r-1}^S + \frac{1}{8(2r-1)!}\int_0^1 f^{(2r-1)}(t)F_{2r-1}^S(t)dt.$$

If $n = 2r$, $r \geq 2$, then

$$\int_0^1 f(t)dt = \frac{1}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right] + T_{r-1}^S + \frac{1}{8(2r)!}\int_0^1 f^{(2r)}(t)F_{2r}^S(t)dt$$

and

$$\int_0^1 f(t)dt = \frac{1}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right] + T_r^S + \frac{1}{8(2r)!}\int_0^1 f^{(2r)}(t)G_{2r}^S(t)dt,$$

where $T_0^S(f) = 0$ and for $1 \leq m \leq \frac{n}{2}$

$$T_m^S(f) = \sum_{k=1}^{m} \frac{1}{8(2k)!}(1 - 3^{2-2k})B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]$$

and

$$G_n^S = 2B_n^*(1-t) + 3B_n^*\left(\frac{1}{3} - t\right) + 3B_n^*\left(\frac{2}{3} - t\right),$$

$$F_n^S = 2B_n^*(1-t) + 3B_n^*\left(\frac{1}{3} - t\right) + 3B_n^*\left(\frac{2}{3} - t\right) - 2B_n - 3B_n\left(\frac{1}{3}\right) - 3B_n\left(\frac{2}{3}\right).$$

It was proved in [8] that $F_n^S(1-t)=(-1)^n F_n^S(t)$ and that $(-1)^r F_{2r-1}^S(t)\geq 0$ in $\left[0,\frac{1}{2}\right]$. Also

$$\int_0^1 \left|F_{2r-1}^S(t)\right| dt = \frac{2}{r}(1-2^{-2r})(1-3^{2-2r})|B_{2r}|,$$

$$\int_0^1 \left|F_{2r}^S(t)\right| dt = (1-3^{2-2r})|B_{2r}|$$

and

$$\int_0^1 \left|G_{2r}^S(t)\right| dt \leq 2(1-3^{2-2r})|B_{2r}|.$$

We can parallel the development of the second section with the following theorem.

Theorem 6. *Suppose $f:[0,1]\to\mathbf{R}$ is n-times differentiable.*
(a) *If $\left|f^{(n)}\right|^q$ is convex for some $q\geq 1$, then for $n=2r-1$, $r\geq 2$, we have*

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[f(0)+3f\left(\frac{1}{3}\right)+3f\left(\frac{2}{3}\right)+f(1)\right]-T_{r-1}^S\right|$$
$$\leq \frac{|B_{2r}|}{2(2r)!}(1-2^{-2r})(1-3^{2-2r})\left[\frac{\left|f^{(2r-1)}(0)\right|^q+\left|f^{(2r-1)}(1)\right|^q}{2}\right]^{1/q}.$$

If $n=2r$, $r\geq 2$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[f(0)+3f\left(\frac{1}{3}\right)+3f\left(\frac{2}{3}\right)+f(1)\right]-T_{r-1}^S\right|$$
$$\leq \frac{|B_{2r}|}{8(2r)!}(1-3^{2-2r})\left[\frac{\left|f^{(2r)}(0)\right|^q+\left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q}$$

and also, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[f(0)+3f\left(\frac{1}{3}\right)+3f\left(\frac{2}{3}\right)+f(1)\right]-T_{r}^S\right|$$
$$\leq \frac{|B_{2r}|}{4(2r)!}(1-3^{2-2r})\left[\frac{\left|f^{(2r)}(0)\right|^q+\left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q}.$$

(b) *If $\left|f^{(n)}\right|$ is concave, then then for $n=2r-1$, $r\geq 2$, we have*

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[f(0)+3f\left(\frac{1}{3}\right)+3f\left(\frac{2}{3}\right)+f(1)\right]-T_{r-1}^S\right|$$
$$\leq \frac{|B_{2r}|}{2(2r)!}(1-2^{-2r})(1-3^{2-2r})\left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|.$$

If $n = 2r,\ r \geq 2,$ *then*

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right] - T_{r-1}^S\right|$$
$$\leq \frac{|B_{2r}|}{8(2r)!}(1 - 3^{2-2r})\left|f^{(2r)}\left(\frac{1}{2}\right)\right|$$

and also, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right] - T_r^S\right|$$
$$\leq \frac{|B_{2r}|}{4(2r)!}(1 - 3^{2-2r})\left|f^{(2r)}\left(\frac{1}{2}\right)\right|.$$

The resultant formulae in Theorem 6 when $n = 3$ are of special interest and we isolate them as corollary.

Corollary 6. *Suppose* $f : [0,1] \to \mathbf{R}$ *is 3-times differentiable.*
(a) *If* $\left|f^{(3)}\right|^q$ *is convex for some* $q \geq 1$, *then*

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[f(0)+3f\left(\frac{1}{3}\right)+3f\left(\frac{2}{3}\right)+f(1)\right]\right| \leq \frac{1}{1728}\left[\frac{|f^{(3)}(0)|^q + |f^{(3)}(1)|^q}{2}\right]^{\frac{1}{q}}.$$

(b) *If* $\left|f^{(3)}\right|$ *is concave, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right]\right| \leq \frac{1}{1728}\left|f^{(3)}\left(\frac{1}{2}\right)\right|.$$

8. The Euler-Maclaurin formulae

In this section we explore a path that is associated with the Euler-Maclaurin formulae.

If f is defined on segment $[0,1]$ and has n ($n \geq 3$) continuous derivatives there, then the Euler-Maclaurin formulae (see [9]) state that if $n = 2r - 1,\ r \geq 2$

$$\int_0^1 f(t)dt = \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] - T_{r-1}^L(f)$$
$$+ \frac{1}{8(2r-1)!}\int_0^1 f^{(2r-1)}(t)F_{2r-1}^L(t)dt.$$

If $n = 2r,\ r \geq 2$, then

$$\int_0^1 f(t)dt = \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] - T_{r-1}^L(f) + \frac{1}{8(2r)!}\int_0^1 f^{(2r)}(t)F_{2r}^L(t)dt$$

and

$$\int_0^1 f(t)dt = \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] - T_{r}^L(f) + \frac{1}{8(2r)!}\int_0^1 f^{(2r)}(t)G_{2r}^L(t)dt,$$

where $T_0^L(f) = 0$ and for $1 \leq m \leq \frac{n}{2}$

$$T_m^L(f) = \sum_{k=1}^{m} \frac{1}{8(2k)!}(1 - 2^{1-2k})(1 - 3^{2-2k})B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]$$

and

$$G_n^L = 3B_n^*\left(\frac{1}{6} - t\right) + 2B_n^*\left(\frac{1}{2} - t\right) + 3B_n^*\left(\frac{5}{6} - t\right),$$

$$F_n^L = 3B_n^*\left(\frac{1}{6} - t\right) + 2B_n^*\left(\frac{1}{2} - t\right) + 3B_n^*\left(\frac{5}{6} - t\right) - 3B_n\left(\frac{1}{6}\right) - 2B_n\left(\frac{1}{2}\right) - 3B_n\left(\frac{5}{6}\right).$$

It was proved in [9] that $F_n^L(1-t) = (-1)^n F_n^L(t)$ and $(-1)^{r-1}F_{2r-1}^L(t) \geq 0$ in $\left[0, \frac{1}{2}\right]$. Also

$$\int_0^1 \left|F_{2r-1}^L(t)\right| dt = \frac{1}{r}(2 - 2^{1-2r})(1 - 3^{2-2r})|B_{2r}|,$$

$$\int_0^1 \left|F_{2r}^L(t)\right| dt = (1 - 2^{1-2r})(1 - 3^{2-2r})|B_{2r}|$$

and

$$\int_0^1 \left|G_{2r}^L(t)\right| dt \leq 2(1 - 2^{1-2r})(1 - 3^{2-2r})|B_{2r}|.$$

We can parallel the development of the second section with the following theorem.

Theorem 7. Suppose $f : [0,1] \to \mathbf{R}$ is n-times differentiable.

(a) If $\left|f^{(n)}\right|^q$ is convex for some $q \geq 1$, then for $n = 2r-1,\ r \geq 2$, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] + T_{r-1}^L(f)\right|$$
$$\leq \frac{|B_{2r}|}{4(2r)!}(2 - 2^{1-2r})(1 - 3^{2-2r})\left[\frac{\left|f^{(2r-1)}(0)\right|^q + \left|f^{(2r-1)}(1)\right|^q}{2}\right]^{1/q}.$$

If $n = 2r,\ r \geq 2$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] + T_{r-1}^L(f)\right|$$
$$\leq \frac{|B_{2r}|}{8(2r)!}(1 - 2^{1-2r})(1 - 3^{2-2r})\left[\frac{\left|f^{(2r)}(0)\right|^q + \left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q}$$

and also, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] + T_r^L(f)\right|$$
$$\leq \frac{|B_{2r}|}{4(2r)!}(1 - 2^{1-2r})(1 - 3^{2-2r})\left[\frac{\left|f^{(2r)}(0)\right|^q + \left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q}.$$

(b) If $\left|f^{(2r)}\right|$ is concave, then for $n = 2r-1,\ r \geq 2$, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] + T_{r-1}^L(f)\right|$$
$$\leq \frac{|B_{2r}|}{4(2r)!}(2 - 2^{1-2r})(1 - 3^{2-2r})\left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|.$$

If $n = 2r,\ r \geq 2$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] + T_{r-1}^L(f)\right|$$
$$\leq \frac{|B_{2r}|}{8(2r)!}(1 - 2^{1-2r})(1 - 3^{2-2r})\left|f^{(2r)}\left(\frac{1}{2}\right)\right|$$

and also, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[3f\left(\frac{1}{6}\right) + 2f\left(\frac{1}{2}\right) + 3f\left(\frac{5}{6}\right)\right] + T_r^L(f)\right|$$
$$\leq \frac{|B_{2r}|}{4(2r)!}(1 - 2^{1-2r})(1 - 3^{2-2r})\left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|.$$

The resultant formulae in Theorem 7 when $n = 3$ are of special interest and we isolate them as corollary.

Corollary 7. *Suppose $f : [0,1] \to \mathbf{R}$ is 3-times differentiable.*
(a) *If $|f^{(3)}|^q$ is convex for some $q \geq 1$, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[3f\left(\frac{1}{6}\right)+2f\left(\frac{1}{2}\right)+3f\left(\frac{5}{6}\right)\right]\right| \leq \frac{1}{1728}\left[\frac{|f^{(3)}(0)|^q+|f^{(3)}(1)|^q}{2}\right]^{\frac{1}{q}}.$$

(b) *If $|f^{(3)}|$ is concave, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{8}\left[3f\left(\frac{1}{6}\right)+2f\left(\frac{1}{2}\right)+3f\left(\frac{5}{6}\right)\right]\right| \leq \frac{1}{1728}\left|f^{(3)}\left(\frac{1}{2}\right)\right|.$$

9. The Euler-Bool formulae

In this section we explore a path that is associated with the Euler-Bool formulae.

If f is defined on segment $[0,1]$ and has n ($n \geq 5$) continuous derivatives there, then the Euler-Bool formulae (see [14]) state that if $n = 2r-1$, $r \geq 3$

$$\int_0^1 f(t)dt = \frac{1}{90}\left[7f(0)+32f\left(\frac{1}{4}\right)+12f\left(\frac{1}{2}\right)+32f\left(\frac{3}{4}\right)+7f(1)\right] - T_{r-1}^B(f) + \frac{1}{90(2r-1)!}\int_0^1 f^{(2r-1)}(t)F_{2r-1}^B(t)dt.$$

If $n = 2r$, $r \geq 3$, then

$$\int_0^1 f(t)dt = \frac{1}{90}\left[7f(0)+32f\left(\frac{1}{4}\right)+12f\left(\frac{1}{2}\right)+32f\left(\frac{3}{4}\right)+7f(1)\right] - T_{r-1}^B(f) + \frac{1}{90(2r)!}\int_0^1 f^{(2r)}(t)F_{2r}^B(t)dt$$

and

$$\int_0^1 f(t)dt = \frac{1}{90}\left[7f(0)+32f\left(\frac{1}{4}\right)+12f\left(\frac{1}{2}\right)+32f\left(\frac{3}{4}\right)+7f(1)\right] - T_r^B(f) + \frac{1}{90(2r)!}\int_0^1 f^{(2r)}(t)G_{2r}^B(t)dt,$$

where $T_0^B(f) = 0$ and for $1 \le m \le \frac{n}{2}$

$$T_m^B(f) = \sum_{k=1}^{m} \frac{1}{90(2k)!}(2 - 5 \cdot 2^{3-2k} + 2^{7-4k})B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]$$

and

$$G_n^B = 14B_n^*(1-t) + 32B_n^*\left(\frac{1}{4}-t\right) + 12B_n^*\left(\frac{1}{2}-t\right) + 32B_n^*\left(\frac{3}{4}-t\right),$$

$$F_n^B = 14B_n^*(1-t) + 32B_n^*\left(\frac{1}{4}-t\right) + 12B_n^*\left(\frac{1}{2}-t\right) + 32B_n^*\left(\frac{3}{4}-t\right)$$
$$- 14B_n - 32B_n\left(\frac{1}{4}\right) - 12B_n\left(\frac{1}{2}\right) - 32B_n^*\left(\frac{3}{4}\right).$$

It was proved in [14] that $F_n^B(1-t) = (-1)^n F_n^B(t)$ and $(-1)^{r-1}F_{2r-1}^B(t) \ge 0$ in $\left[0, \frac{1}{2}\right]$. Also

$$\int_0^1 \left|F_{2r-1}^B(t)\right| dt = \frac{4}{r}(1 - 2^{-2r})|B_{2r}|,$$

$$\int_0^1 \left|F_{2r}^B(t)\right| dt = (2 - 5 \cdot 2^{3-2r} + 2^{7-4r})|B_{2r}|$$

and

$$\int_0^1 \left|G_{2r}^B(t)\right| dt \le 2(2 - 5 \cdot 2^{3-2r} + 2^{7-4r})|B_{2r}|.$$

We can parallel the development of the second section with the following theorem.

Theorem 8. *Suppose $f : [0,1] \to \mathbf{R}$ is n-times differentiable.*
(a) *If $\left|f^{(n)}\right|^q$ is convex for some $q \ge 1$, then for $n = 2r-1$, $r \ge 3$, we have*

$$\left|\int_0^1 f(t)dt - \frac{1}{90}\left[7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1)\right] + T_{r-1}^B(f)\right|$$
$$\le \frac{|B_{2r}|}{45(2r)!}4(1 - 2^{-2r})\left[\frac{\left|f^{(2r-1)}(0)\right|^q + \left|f^{(2r-1)}(1)\right|^q}{2}\right]^{1/q}.$$

If $n = 2r$, $r \ge 3$, then

$$\left|\int_0^1 f(t)dt - \frac{1}{90}\left[7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1)\right] + T_{r-1}^B(f)\right|$$
$$\le \frac{|B_{2r}|}{45(2r)!}(1 - 5 \cdot 2^{2-2r} + 2^{6-4r})\left[\frac{\left|f^{(2r)}(0)\right|^q + \left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q}$$

and also, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{90}\left[7f(0)+32f\left(\frac{1}{4}\right)+12f\left(\frac{1}{2}\right)+32f\left(\frac{3}{4}\right)+7f(1)\right]+T_r^B(f)\right|$$
$$\leq \frac{|B_{2r}|}{45(2r)!}(2-5\cdot 2^{3-2r}+2^{7-4r})\left[\frac{\left|f^{(2r)}(0)\right|^q+\left|f^{(2r)}(1)\right|^q}{2}\right]^{1/q}.$$

(b) *If* $\left|f^{(2r)}\right|$ *is concave, then for* $n=2r-1,\ r\geq 3$, *we have*

$$\left|\int_0^1 f(t)dt - \frac{1}{90}\left[7f(0)+32f\left(\frac{1}{4}\right)+12f\left(\frac{1}{2}\right)+32f\left(\frac{3}{4}\right)+7f(1)\right]+T_{r-1}^B(f)\right|$$
$$\leq \frac{|B_{2r}|}{45(2r)!}4(1-2^{-2r})\left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|.$$

If $n=2r,\ r\geq 3$, *then*

$$\left|\int_0^1 f(t)dt - \frac{1}{90}\left[7f(0)+32f\left(\frac{1}{4}\right)+12f\left(\frac{1}{2}\right)+32f\left(\frac{3}{4}\right)+7f(1)\right]+T_{r-1}^B(f)\right|$$
$$\leq \frac{|B_{2r}|}{45(2r)!}(1-5\cdot 2^{2-2r}+2^{6-4r})\left|f^{(2r)}\left(\frac{1}{2}\right)\right|$$

and also, we have

$$\left|\int_0^1 f(t)dt - \frac{1}{90}\left[7f(0)+32f\left(\frac{1}{4}\right)+12f\left(\frac{1}{2}\right)+32f\left(\frac{3}{4}\right)+7f(1)\right]+T_r^B(f)\right|$$
$$\leq \frac{|B_{2r}|}{45(2r)!}(2-5\cdot 2^{3-2r}+2^{7-4r})\left|f^{(2r-1)}\left(\frac{1}{2}\right)\right|.$$

The resultant formulae in Theorem 8 when $n=5$ are of special interest and we isolate them as corollary.

Corollary 8. *Suppose* $f:[0,1]\to\mathbf{R}$ *is* 5*-times differentiable.*
(a) *If* $\left|f^{(5)}\right|^q$ *is convex for some* $q\geq 1$, *then*

$$\left|\int_0^1 f(t)dt - \frac{1}{90}\left[7f(0)+32f\left(\frac{1}{4}\right)+12f\left(\frac{1}{2}\right)+32f\left(\frac{3}{4}\right)+7f(1)\right]\right|$$
$$\leq \frac{1}{345600}\left[\frac{\left|f^{(5)}(0)\right|^q+\left|f^{(5)}(1)\right|^q}{2}\right]^{1/q}.$$

(b) *If* $\left|f^{(5)}\right|$ *is concave, then*

$$\left|\int_0^1 f(t)dt - \frac{1}{90}\left[7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1)\right]\right| \leq \frac{1}{345600}\left|f^{(5)}\left(\frac{1}{2}\right)\right|.$$

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ГЕНЕРЕЛИЗАЦИИ НА НЕРАВЕНСТВАТА НА ДРАГОМИР-АГАРВАЛ СО ПРИМЕНА НА НЕКОИ ИДЕНТИТЕТИ ОД ТИПОТ НА ОЈЛЕР

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Резиме

Дадени се неколку генерализации на неравенството на Драгомир-Аагарвал со примена на некои идентитети од типот на Ојлер.

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