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ON PEČARIĆ'S INEQUALITY IN INNER PRODUCT SPACES

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Abstract

Some related results to Pečarić's inequality in inner product spaces that generalises Bombieri's inequality, are given.

1. Introduction

In 1992, J. E. Pečarić [3] proved the following inequality for vectors in complex inner product spaces $(H;\ (\cdot,\cdot))$.

Theorem 1. *Suppose that $x, y_1, \ldots, y_n$ are vectors in H and $c_1, \ldots, c_n$ are complex numbers. Then the following inequalities*

$$\left|\sum_{i=1}^{n} c_i(x, y_i)\right|^2 \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \Big(\sum_{j=1}^{n} |(y_i, y_j)|\Big) \tag{1.1}$$

$$\leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \max_{1\leq i\leq n} \Big(\sum_{j=1}^{n} |(y_i, y_j)|\Big),$$

hold.

He also showed that for $c_i = \overline{(x, y_i)}, i \in \{1, \ldots, n\}$, one gets

$$\left(\sum_{i=1}^{n}|(x,y_i)|^2\right)^2 \le \|x\|^2\sum_{i=1}^{n}|(x,y_i)|^2\left(\sum_{j=1}^{n}|(y_i,y_j)|\right) \tag{1.2}$$

$$\le \|x\|^2\sum_{i=1}^{n}|(x,y_i)|^2 \max_{1\le i\le n}\left(\sum_{j=1}^{n}|(y_i,y_j)|\right),$$

which improves Bombieri's result [1] (see also [2,p. 394])

$$\sum_{i=1}^{n}|(x,y_i)|^2 \le \|x\|^2 \max_{1\le i\le n}\left(\sum_{j=1}^{n}|(y_i,y_j)|\right). \tag{1.3}$$

Note that (1.3) is in its turn a natural generalisation of *Bessel's inequality*

$$\sum_{i=1}^{n}|(x,e_i)|^2 \le \|x\|^2, \quad x\in H, \tag{1.4}$$

which holds for the orthonormal vectors $(e_i)_{1\le i\le n}$.

In this paper we point out some related results to Pečarić's inequality (1.1). Some results of Bombieri type are also mentioned.

2. Preliminary results

We start with the following lemma that is interesting in its own right.

Lemma 1. *Let $z_1,\dots,z_n\in H$ and $\alpha_1,\dots,\alpha_n\in\mathbf{K}$. Then one has the inequalities:*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \le \tag{2.1}$$

$$\le \left(\sum_{i=1}^{n}|\alpha_i|^p\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)\right)^{1/p}\left(\sum_{i=1}^{n}|\alpha_i|^q\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)\right)^{1/q}$$

$$
\le \begin{cases}
\max\limits_{1\le i\le n} |\alpha_i|^2 \sum\limits_{i,j=1}^{n} |(z_i, z_j)|\,; \\[2ex]
\max\limits_{1\le i\le n} |\alpha_i| \Big(\sum\limits_{i=1}^{n} |\alpha_i|^{\gamma q}\Big)^{1/\gamma q} \Big(\sum\limits_{i,j=1}^{n} |(z_i, z_j)|\Big)^{1/p} \\[1ex]
\quad \times \Big(\sum\limits_{i=1}^{n}\Big(\sum\limits_{j=1}^{n} |(z_i, z_j)|\Big)^{\delta}\Big)^{1/\delta q} \qquad \text{if} \quad \gamma > 1, \ \dfrac{1}{\gamma} + \dfrac{1}{\delta} = 1\,; \\[2ex]
\max\limits_{1\le i\le n} |\alpha_i| \Big(\sum\limits_{i=1}^{n} |\alpha_i|^{q}\Big)^{1/q} \Big(\sum\limits_{i,j=1}^{n} |(z_i, z_j)|\Big)^{1/p} \max\limits_{1\le i\le n} \Big(\sum\limits_{j=1}^{n} |(z_i, z_j)|\Big)^{1/q}\,; \\[2ex]
\max\limits_{1\le i\le n} |\alpha_i| \Big(\sum\limits_{i=1}^{n} |\alpha_i|^{\alpha p}\Big)^{1/\alpha p} \Big(\sum\limits_{i,j=1}^{n} |(z_i, z_j)|\Big)^{1/q} \\[1ex]
\quad \times \Big(\sum\limits_{i=1}^{n}\Big(\sum\limits_{j=1}^{n} |(z_i, z_j)|\Big)^{\beta}\Big)^{1/\beta q} \quad \text{if} \quad \alpha > 1, \ \dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1\,; \\[2ex]
\Big(\sum\limits_{i=1}^{n} |\alpha_i|^{\alpha p}\Big)^{1/\alpha p} \Big(\sum\limits_{i=1}^{n} |\alpha_i|^{\gamma q}\Big)^{1/\gamma q} \Big(\sum\limits_{i=1}^{n}\Big(\sum\limits_{j=1}^{n} |(z_i, z_j)|\Big)^{\beta}\Big)^{1/p\beta} \\[1ex]
\quad \times \Big(\sum\limits_{i=1}^{n}\Big(\sum\limits_{j=1}^{n} |(z_i, z_j)|\Big)^{\delta}\Big)^{1/\delta q} \\
\qquad\qquad \text{if} \quad \alpha > 1, \ \dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1 \quad \text{and} \quad \gamma > 1, \ \dfrac{1}{\gamma} + \dfrac{1}{\delta} = 1\,; \\[2ex]
\Big(\sum\limits_{i=1}^{n} |\alpha_i|^{q}\Big)^{1/q} \Big(\sum\limits_{i=1}^{n} |\alpha_i|^{\alpha p}\Big)^{1/\alpha p} \max\limits_{1\le i\le n} \Big(\sum\limits_{j=1}^{n} |(z_i, z_j)|\Big)^{1/q} \\[1ex]
\quad \times \Big(\sum\limits_{i=1}^{n}\Big(\sum\limits_{j=1}^{n} |(z_i, z_j)|\Big)^{\beta}\Big)^{1/p\beta}, \quad \text{if} \quad \gamma > 1, \ \dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1\,; \\[2ex]
\max\limits_{1\le i\le n} |\alpha_i| \Big(\sum\limits_{i=1}^{n} |\alpha_i|^{p}\Big)^{1/p} \max\limits_{1\le i\le n} \Big(\sum\limits_{j=1}^{n} |(z_i, z_j)|\Big)^{1/p} \Big(\sum\limits_{i,j=1}^{n} |(z_i, z_j)|\Big)^{1/q}\,; \\[2ex]
\Big(\sum\limits_{i=1}^{n} |\alpha_i|^{p}\Big)^{1/p} \Big(\sum\limits_{i=1}^{n} |\alpha_i|^{\gamma q}\Big)^{1/\gamma q} \max\limits_{1\le i\le n} \Big(\sum\limits_{j=1}^{n} |(z_i, z_j)|\Big)^{1/p} \\[1ex]
\quad \times \Big(\sum\limits_{i=1}^{n}\Big(\sum\limits_{j=1}^{n} |(z_i, z_j)|\Big)^{\delta}\Big)^{1/\delta q}, \quad \text{if} \quad \gamma > 1, \ \dfrac{1}{\gamma} + \dfrac{1}{\delta} = 1\,; \\[2ex]
\Big(\sum\limits_{i=1}^{n} |\alpha_i|^{p}\Big)^{1/p} \Big(\sum\limits_{i=1}^{n} |\alpha_i|^{q}\Big)^{1/q} \max\limits_{1\le i\le n} \Big(\sum\limits_{j=1}^{n} |(z_i, z_j)|\Big),
\end{cases}
$$

where $p > 1,\ \ \frac{1}{p} + \frac{1}{q} = 1.$

Proof.

$$\Big\|\sum_{i=1}^{n} \alpha_i z_i\Big\|^2 = \Big(\sum_{i=1}^{n} \alpha_i z_i \sum_{j=1}^{n} \alpha_j z_j\Big) \tag{2.2}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\alpha}_j (z_i, z_j) \Big| \sum_{i=1}^{n}$$

$$= \sum_{j=1}^{n} \alpha_i \overline{\alpha}_j (z_i, z_j) \Big|$$

$$\leq \sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha_i\ |\alpha_j|\ |(z_i, z_j)| =: M.$$

If one uses the Hölder inequality for double sums, i.e., we recall it

$$\sum_{i,j=1}^{n} m_{ij} a_{ij} b_{ij} \leq \Big(\sum_{i,j=1}^{n} m_{ij} a_{ij}^{p}\Big)^{1/p} \Big(\sum_{i,j=1}^{n} m_{ij} b_{ij}^{q}\Big)^{1/q} \tag{2.3}$$

where $m_{ij}, a_{ij}, b_{ij} \geq 0,\ \ \frac{1}{p} + \frac{1}{q} = 1,\ \ p > 1$; then

$$M \leq \Big(\sum_{i,j=1}^{n} |(z_i, z_j)|\ |\alpha_i|^p\Big)^{1/p} \Big(\sum_{i,j=1}^{n} |(z_i, z_j)|\ |\alpha_i|^q\Big)^{1/q} \tag{2.4}$$

$$= \Big(\sum_{i=1}^{n} |\alpha_i|^p \Big(\sum_{j=1}^{n} |(z_i, z_j)|\Big)\Big)^{1/p} \Big(\sum_{i=1}^{n} |\alpha_i|^q \Big(\sum_{j=1}^{n} |(z_i, z_j)|\Big)\Big)^{1/q},$$

and the first inequality in (2.1) is proved.

Observe that

$$\sum_{i=1}^{n}|\alpha_i|^p\Big(\sum_{j=1}^{n}|(z_i,z_j)|\Big)\le$$

$$\le\begin{cases}\max\limits_{1\le i\le n}|\alpha_i|^p\sum\limits_{i,j=1}^{n}|(z_i,z_j)|\,; \\ \Big(\sum\limits_{i=1}^{n}|\alpha_i|^{\alpha p}\Big)^{1/\alpha}\Big(\sum\limits_{i=1}^{n}\Big(\sum\limits_{j=1}^{n}|(z_i,z_j)|\Big)^{\beta}\Big)^{1/\beta} \\ \qquad \text{if } \alpha>1,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1\,; \\ \sum\limits_{i=1}^{n}|\alpha_i|^p\max\limits_{1\le i\le n}\Big(\sum\limits_{j=1}^{n}|(z_i,z_j)|\Big)\,;\end{cases}$$

giving

$$\Big(\sum_{i=1}^{n}|\alpha_i|^p\Big(\sum_{j=1}^{n}|(z_i,z_j)|\Big)\Big)^{1/p}\le \tag{2.5}$$

$$\le\begin{cases}\max\limits_{1\le i\le n}|\alpha_i|\Big(\sum\limits_{i,j=1}^{n}|(z_i,z_j)|\Big)^{1/p}\,; \\ \Big(\sum\limits_{i=1}^{n}|\alpha_i|^{\alpha p}\Big)^{1/\alpha p}\Big(\sum\limits_{i=1}^{n}\Big(\sum\limits_{j=1}^{n}|(z_i,z_j)|\Big)^{\beta}\Big)^{1/\beta p} \\ \qquad \text{if } \alpha>1,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1\,; \\ \Big(\sum\limits_{i=1}^{n}|\alpha_i|^p\Big)^{1/p}\max\limits_{1\le i\le n}\Big(\sum\limits_{j=1}^{n}|(z_i,z_j)|\Big)^{1/p}.\end{cases}$$

Similarly, we have

$$\left(\sum_{i=1}^{n}|\alpha_i|^q\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)\right)^{1/q}\le \tag{2.6}$$

$$\le\begin{cases}\max\limits_{1\le i\le n}|\alpha_i|\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j)|\right)^{1/q};\\[2ex] \left(\sum\limits_{i=1}^{n}|\alpha_i|^{\gamma q}\right)^{1/\gamma q}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\delta}\right)^{1/\delta q}\\[2ex] \qquad\text{if } \gamma>1,\ \dfrac{1}{\gamma}+\dfrac{1}{\delta}=1;\\[2ex] \left(\sum\limits_{i=1}^{n}|\alpha_i|^q\right)^{1/q}\max\limits_{1\le i\le n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{1/q}.\end{cases}$$

Using (2.1) and (2.5) – (2.6), we deduce the 9 inequalities in the second part of (2.2). □

If we choose $p=q=2$, then the following result holds.

Corollary 1. *If $z_1,\dots,z_n\in H$ and $\alpha_1,\dots,\alpha_n\in\mathbf{K}$, then one has*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2\le \tag{2.7}$$

$$\le\sum_{i=1}^{n}|\alpha_i|^2\left(\sum_{j=1}^{n}|(z_i,z_j)|\right)$$

$$\leq \begin{cases} \max\limits_{1\leq i\leq n}|\alpha_i|^2 \sum\limits_{i,j=1}^{n}|(z_i,z_j)|\,; \\ \max\limits_{1\leq i\leq n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\gamma}\right)^{1/2\gamma}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j)|\right)^{1/2}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\delta}\right)^{1/2\delta}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1\,; \\ \max\limits_{1\leq i\leq n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2}\right)^{1/2}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j)|\right)^{1/2}\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{1/2}; \\ \max\limits_{1\leq i\leq n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\alpha}\right)^{1/2\alpha}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j)|\right)^{1/2}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\beta}\right)^{1/2\beta} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1\,; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\alpha}\right)^{1/2\alpha}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\gamma}\right)^{1/2\gamma}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\beta}\right)^{1/2\beta} \\ \quad\times\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\delta}\right)^{1/2\delta} \\ \qquad\qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \text{ and } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1\,; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^{2}\right)^{1/2}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\alpha}\right)^{1/2\alpha}\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{1/2} \\ \quad\times\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\beta}\right)^{1/2\beta}, \qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1\,; \\ \max\limits_{1\leq i\leq n}|\alpha_i|\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2}\right)^{1/2}\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{1/2}\left(\sum\limits_{i,j=1}^{n}|(z_i,z_j)|\right)^{1/2}; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^{2}\right)^{1/2}\left(\sum\limits_{i=1}^{n}|\alpha_i|^{2\gamma}\right)^{1/2\gamma}\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{1/2} \\ \quad\times\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^{\delta}\right)^{1/2\delta}, \qquad \text{if } \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1\,; \\ \sum\limits_{i=1}^{n}|\alpha_i|^2\max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right). \end{cases}$$

3. Some Pečarić Type Inequalities

We are now able to point out the following result which complements and generalises the inequality (1.1) due to J. Pečarić.

Theorem 2. *Let $x, y_1, \ldots, y_n$ be vectors of an inner product space $(H; (\cdot,\cdot)$ and $c_1, \ldots, c_n \in \mathbf{K}$. Then one has the inequalities:*

$$\left|\sum_{i=1}^{n} c_i(x, y_i)\right|^2 \le \tag{3.1}$$
$$\le \|x\|^2 \left(\sum_{i=1}^{n} |c_i|^p \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)\right)^{1/p} \left(\sum_{i=1}^{n} |c_i|^q \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)\right)^{1/q}$$

$$\le \|x\|^2 \times \begin{cases} \max\limits_{1\le i\le n} |c_i|^2 \sum\limits_{i,j=1}^{n} |(y_i, y_j)|; \\ \max\limits_{1\le i\le n} |c_i| \left(\sum\limits_{i=1}^{n} |c_i|^{\gamma q}\right)^{1/\gamma q} \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j)|\right)^{1/p} \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right)^{\delta}\right)^{1/\delta q}, \\ \qquad \text{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \max\limits_{1\le i\le n} |c_i| \left(\sum\limits_{i=1}^{n} |c_i|^{q}\right)^{1/q} \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j)|\right)^{1/p} \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right)^{1/q}; \\ \max\limits_{1\le i\le n} |c_i| \left(\sum\limits_{i=1}^{n} |c_i|^{\alpha p}\right)^{1/\alpha p} \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j)|\right)^{1/q} \\ \quad \times \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right)^{\beta}\right)^{1/p\beta}, \quad \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \left(\sum\limits_{i=1}^{n} |c_i|^{\alpha p}\right)^{1/\alpha p} \left(\sum\limits_{i=1}^{n} |c_i|^{\gamma q}\right)^{1/\gamma q} \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right)^{\beta}\right)^{1/p\beta} \\ \quad \times \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right)^{\delta}\right)^{1/\delta q} \\ \qquad \text{if } \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1 \text{ and } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left(\sum\limits_{i=1}^{n} |c_i|^{q}\right)^{1/q} \left(\sum\limits_{i=1}^{n} |c_i|^{\alpha p}\right)^{1/\alpha p} \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right)^{1/q} \\ \quad \times \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right)^{\beta}\right)^{1/p\beta}, \quad \text{if } \alpha > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \max\limits_{1\le i\le n} |c_i| \left(\sum\limits_{i=1}^{n} |c_i|^{p}\right)^{1/p} \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right)^{1/p} \left(\sum\limits_{i,j=1}^{n} |(y_i, y_j)|\right)^{1/q}; \\ \left(\sum\limits_{i=1}^{n} |c_i|^{p}\right)^{1/p} \left(\sum\limits_{i=1}^{n} |c_i|^{\gamma q}\right)^{1/\gamma q} \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right)^{1/p} \\ \quad \times \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right)^{\delta}\right)^{1/\delta q}, \quad \text{if } \gamma > 1,\ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \left(\sum\limits_{i=1}^{n} |c_i|^{p}\right)^{1/p} \left(\sum\limits_{i=1}^{n} |c_i|^{q}\right)^{1/q} \max\limits_{1\le i\le n} \left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right); \end{cases}$$

where $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1.$

Proof. We note that

$$\sum_{i=1}^{n} c_i(x, y_i) = \left(x, \sum_{i=1}^{n} \overline{c}_i y_i\right).$$

Using Schwarz's inequality in inner product spaces, we have

$$\left|\sum_{i=1}^{n} c_i(x, y_i)\right|^2 \le \|x\|^2 \left\|\sum_{i=1}^{n} \overline{c}_i y_i\right\|^2. \tag{3.2}$$

Finally, using Lemma 1 with $\alpha_i = \overline{c}_i,\ z_i = y_i\ (i = 1, \ldots, n)$, we deduce the desired inequality (3.1). □

Remark 1. *If in* (3.1) *we choose* $p = q = 2$, *we obtain amongst others, the result* (1.1) *due to J. Pečarić.*

4. Some Results of Bombieri Type

The following results of Bombieri type hold.

Theorem 3. *Let* $x, y_1, \ldots, y_n \in H$. *Then one has the inequality:*

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le \tag{4.1}$$

$$\le \|x\| \left[\sum_{i=1}^{n} |(x, y_i)|^p \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)\right]^{1/2p}$$

$$\times \left[\sum_{i=1}^{n} |(x, y_i)|^q \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)\right]^{1/2q}$$

$$
\leq \|x\| \times
\begin{cases}
\max\limits_{1\leq i\leq n}|(x,y_i)|\Big(\sum\limits_{i,j=1}^{n}|(y_i,y_j)|\Big)^{1/2}; \\
\max\limits_{1\leq i\leq n}|(x,y_i)|^{1/2}\Big(\sum\limits_{i=1}^{n}|(x,y_i)|^{\gamma q}\Big)^{1/2\gamma q}\Big(\sum\limits_{i,j=1}^{n}|(y_i,y_j)|\Big)^{1/2p} \\
\quad\times\Big(\sum\limits_{i=1}^{n}\Big(\sum\limits_{j=1}^{n}|(y_i,y_j)|\Big)^{\delta}\Big)^{1/2\delta q} \qquad \text{if} \quad \gamma>1,\ \dfrac{1}{\gamma}+\dfrac{1}{\delta}=1; \\
\max\limits_{1\leq i\leq n}|(x,y_i)|^{1/2}\Big(\sum\limits_{i=1}^{n}|(x,y_i)|^{q}\Big)^{1/2q} \\
\quad\times\Big(\sum\limits_{i,j=1}^{n}|(y_i,y_j)|\Big)^{1/2p}\max\limits_{1\leq i\leq n}\Big(\sum\limits_{j=1}^{n}|(y_i,y_j)|\Big)^{1/2q}; \\
\max\limits_{1\leq i\leq n}|(x,y_i)|^{1/2}\Big(\sum\limits_{i=1}^{n}|(x,y_i)|^{\alpha p}\Big)^{1/2\alpha p}\Big(\sum\limits_{i,j=1}^{n}|(y_i,y_j)|\Big)^{1/2q} \\
\quad\times\Big(\sum\limits_{i=1}^{n}\Big(\sum\limits_{j=1}^{n}|(y_i,y_j)|\Big)^{\beta}\Big)^{1/p\beta} \qquad \text{if} \quad \alpha>1,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1; \\
\Big(\sum\limits_{i=1}^{n}|(x,y_i)|^{\alpha p}\Big)^{1/2\alpha p}\Big(\sum\limits_{i=1}^{n}|(x,y_i)|^{\gamma q}\Big)^{1/2\gamma q}\Big(\sum\limits_{i=1}^{n}\Big(\sum\limits_{j=1}^{n}|(y_i,y_j)|\Big)^{\beta}\Big)^{1/2p\beta} \\
\quad\times\Big(\sum\limits_{i=1}^{n}\Big(\sum\limits_{j=1}^{n}|(y_i,y_j)|\Big)^{\delta}\Big)^{1/2\delta q} \text{if}\ \alpha>1,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1 \text{ and } \gamma>1,\ \dfrac{1}{\gamma}+\dfrac{1}{\delta}=1; \\
\Big(\sum\limits_{i=1}^{n}|(x,y_i)|^{q}\Big)^{1/2q}\Big(\sum\limits_{i=1}^{n}|(x,y_i)|^{\alpha p}\Big)^{1/2\alpha p}\max\limits_{1\leq i\leq n}\Big(\sum\limits_{j=1}^{n}|(y_i,y_j)|\Big)^{1/2p} \\
\quad\times\Big(\sum\limits_{i=1}^{n}\Big(\sum\limits_{j=1}^{n}|(y_i,y_j)|\Big)^{\beta}\Big)^{1/2p\beta}, \qquad \text{if} \quad \alpha>1,\ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1; \\
\max\limits_{1\leq i\leq n}|(x,y_i)|^{1/2}\Big(\sum\limits_{i=1}^{n}|(x,y_i)|^{p}\Big)^{1/2p}\max\limits_{1\leq i\leq n}\Big(\sum\limits_{j=1}^{n}|(y_i,y_j)|\Big)^{1/2p} \\
\quad\times\Big(\sum\limits_{i,j=1}^{n}|(y_i,y_j)|\Big)^{1/2q}; \\
\Big(\sum\limits_{i=1}^{n}|(x,y_i)|^{p}\Big)^{1/2p}\Big(\sum\limits_{i=1}^{n}|(x,y_i)|^{\gamma q}\Big)^{1/2\gamma q}\max\limits_{1\leq i\leq n}\Big(\sum\limits_{j=1}^{n}|(y_i,y_j)|\Big)^{1/2p} \\
\quad\times\Big(\sum\limits_{i=1}^{n}\Big(\sum\limits_{j=1}^{n}|(y_i,y_j)|\Big)^{\delta}\Big)^{1/2\delta q}, \qquad \text{if} \quad \gamma>1,\ \dfrac{1}{\gamma}+\dfrac{1}{\delta}=1; \\
\Big(\sum\limits_{i=1}^{n}|(x,y_i)|^{p}\Big)^{1/2p}\Big(\sum\limits_{i=1}^{n}|(x,y_i)|^{q}\Big)^{1/2q}\max\limits_{1\leq i\leq n}\Big(\sum\limits_{j=1}^{n}|(y_i,y_j)|\Big)^{1/2},
\end{cases}
$$

where $p > 1, \ \dfrac{1}{p} + \dfrac{1}{q} = 1.$

Proof. The proof follows by Theorem 2 on choosing $c_i = \overline{(x, y_i)}$, $i \in \{1, \dots, n\}$ and taking the square root in both sides of the inequalities involved. We omit the details. □

Remark 2. *We observe, by the last inequality in* (4.1), *we get*

$$\frac{\left(\sum_{i=1}^{n} |(x,y_i)|^2\right)^2}{\left(\sum_{i=1}^{n} |(x,y_i)|^p\right)^{1/p}\left(\sum_{i=1}^{n} |(x,y_i)|^q\right)^q} \le \|x\|^2 \max_{1\le i\le n}\left(\sum_{j=1}^{n} |(y_i,y_j)|\right),$$

where $p > 1, \ \dfrac{1}{p} + \dfrac{1}{q} = 1.$

If in this inequality we choose $p = q = 2$, *then we recapture Bombieri's result* (1.3).

References

[1] E. Bombieri: *A note on the large sieve,* Acta Arith., **18** (1971), 401-404.

[2] D. S. Mitrinović, J. E. Pečarić and A. M. Fink: *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, 1993.

[3] J. E. Pečarić: *On some classical inequalities in unitary spaces,* Mat. Bilten (Skopje), **16**, (1992), 63-72,

ЗА НЕРАВЕНСТВОТО НА ПЕЧАРИЌ ВО ПРОСТОРИ СО СКАЛАРЕН ПРОИЗВОД

С. С. Драгомир

Резиме

Дадени се некои резултати кои го генерализираат неравенството на Печариќ во простори со скаларен производ, а го генерализираат неравенството на Бомбиери.

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