ЗА ПРИДРУЖЕНИТЕ LEGENDRE-ОВИ СФЕРНИ ФУНКЦИИ

Билтен на Друштвото на математичарите и физичарите од Н Р Македонија, кн. VIII, 1957, 5-18

1. Овие функции се дефинирани со релацијата

$$P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m}$$

и ја задоволуваат диференцијалната равенка

$$\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right] + \left[n(n+1) - \frac{m^2}{1-x^2}\right] u = 0,$$

каде што е $u = P_n^m(\alpha)$

$P_n(x)$ претставува функција на Legendre, а m е произвлен цел позитавен број.

Често пати наместо со релацијата (1), овие функции се дефинирани со[1]

$$P_n^m(x) = (1-x^2)^{m/2} \frac{d^m P_n(x)}{dx^m},$$

којашто релација ќе ја ползуваме ние во натамошните разгледувања.

Спрема тоа овие функции можат да бидат претставени и со формулата

$$(1)\quad P_n^m(x) = \frac{\Gamma(m+n+1)(1-x^2)^{m/2}}{2^m \Gamma(n-m+1)\Gamma(m+1)} F\left(m-n, m+n+1, m+1, \frac{1-x}{2}\right),$$

каде што е $F(a,b,c,x)$ хипергеометриска функција.

Предмет на овој труд е да даде обопштување на релацијата дадена од Bailey [2] за продукт од две придружени Legendre-ови сферни функции и исто така некои нови формули за овие функции. Познато е дека овие функции наоѓаат голема примена во различни проблеми од квантната механика.

Насекаде во разгледувањата зимаме параметарот n да е природен број, што значи дека $P_n(x)$ се полноми на Legendre.

2. Во една поранешна работа [3], ја добивме следната релација

$$(2)\quad F\left(r-m, r+m+1, r+1, \frac{1-x}{2}\right) F\left(s-n, s+n+1, s+1, \frac{1-x}{2}\right)$$

$$= 2^{2r} \frac{(m-r)!\,(n-s)!\,r!}{(m+r)!\,(n+s)!} \sum_{k=0}^{\lambda} \frac{A_{m-k}^{-r} A_{k,-r} A_{n-k}^{-s}}{A_{m+n-r-k}^{s}} \frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1}$$

$$\times {}_4F_3\left[\begin{matrix} r-s, k+r-m, {}^1/_2-s, -k, \\ n-s-k+1, k+r-m-n-s, {}^1/_2+r \end{matrix}\; 1\right]$$

$$\times F\left(r+s-m-n+2k, m+n-r+s-2k+1, s+1, \frac{1-x}{2}\right)$$

каде што е

$$A_m^r=\frac{\left(\frac{1}{2}\right)_m}{(m+r)!},\quad A_{k,r}=\frac{\left(\frac{1}{2}+r\right)_k}{k!},$$

$${}_4F_3\left[\begin{matrix}\alpha,\beta,\gamma,\delta,\\ a,b,c,\end{matrix}x\right]=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k(\gamma)_k(\delta)_k}{(a)_k(b)_k(c)_k}\frac{x_k}{k!},$$

$$(\alpha)_k=\alpha(\alpha+1)\ldots(\alpha+k-1),\ (\alpha)_0=1,$$

$$\lambda=min\,(m-r,n-s).$$

Зимајќи ја во предвид релацијата (1) и претходната, добиваме лесно

$$(3)\qquad (1-x^2)^{-r/2}P_m^r(x)P_n^s(x)=$$

$$=2^r\sum_{k=0}^{\lambda}\frac{A_{m-k}^{-r}A_{k,-r}A_{n-k}^{-s}}{A_{m+n-r-k}^{s}}\frac{(m+n-r-s-2k)!}{(m+n-r+s-2k)!}$$

$$\times\frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1}$$

$$\times{}_4F_3\left[\begin{matrix}r-s,k+r-m,{}^1/_2-s,-k,\\ n+1-s-k,k+r-m-n-s,{}^1/_2+r\end{matrix}1\right]P_{m+n-r-2k}^s(x),$$

којашто релација ни ја дава композицијата на две придружени Legendre-ови сферни функции.

3. Очигледно е дека релацијата (3) претставува обопштување на спомнатата релација дадена од Bailey, за производ на две придружени Legendre-ови сферни функции.

Навистина, ако земеме $r=s$, добиваме

$$(1-x^2)^{-r/2}P_m^r(x)P_n^r(x)=2^r\sum_{k=0}^{\lambda}\frac{A_{m-k}^{-r}A_{k,-r}A_{n-k}^{-s}}{A_{m+n-r-k}^{s}}\frac{(m+n-2r-2k)!}{(m+n-2k)!}$$

$$\times\frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1}P_{m+n-r-2k}^r(x),$$

$$\lambda=min\,(m-r,n-r).$$

Релацијата (3) ни овозможува да добиеме формула за композиција на еден полином на Legendre и придружена Legendre-ова сферна функција.

Ако ставиме $r=0$, добиваме

$$P_m(x)P_n^s(x) = \sum_{k=0}^{\lambda} \frac{A_{m-k}A_k A_{n-k}^{-s}}{A_{m+n-k}^s} \frac{(m+n-s-2k)!}{(m+n+s-2k)!} \frac{2m+2n-4k+1}{2m+2n-2k+1}$$

$$\times {}_4F_3\left[\begin{matrix} -s, k-m, {}^1/_2-s, -k, \\ n+1-s+k, k-m-n-s, {}^1/_2 \end{matrix}\; 1\right] P_{m+n-2k}^s(x),$$

$$\lambda = min\,(m, n-s).$$

Во случај кога е $s=0$, имаме

$$P_m^r(x)P_n(x) = 2^r \sum_{k=0}^{\lambda} \frac{A_{m-k}^{-r} A_{n-k} A_{k,-r}}{A_{m+n-r-k}} \frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1}$$

$$\times {}_4F_3\left[\begin{matrix} r, k+r-m, {}^1/_2, -k, \\ n-k+1, k+r-m-n, {}^1/_2+r \end{matrix}\; 1\right] (1-x^2)^{r/2} P_{m+n-r-2k}(x),$$

$$\lambda = min\,(m-r, n).$$

Формулата на Neumann—Adams исто така лесно се добива од (2), ако се стави $r=s=0$.

4. Поаѓајќи од (3) може да се добие една релација за производ од произволен број такви функции.

Така на пример, за производ од три функции ќе имаме

$$P_l^p(x)\,P_m^r(x)\,P_n^s(x) = 2^{r+s} \sum_{\varkappa=0}^{\alpha} \sum_{\nu=0}^{\beta} B_{m,n,\varkappa}^{r,s} B_{m+n-r-2\varkappa,l,\nu}^{s,p} \tag{4}$$

$$\times (1-x^2)^{\frac{r+s}{2}} P_{l+m+n-r-s-2\varkappa-2\nu}^p(x),$$

каде што е

$$B_{m,n,i}^{r,s} = \frac{A_{m-i}^{-r} A_{i,-r} A_{n-i}^{-s}}{A_{m+n-r-i}^s} \frac{(m+n-r-s-2i)!}{(m+n-r+s-2i)!}$$

$$\times \frac{2m+2n-2r-4i+1}{2m+2n-2r-2i+1}\; {}_4F_3\left[\begin{matrix} r-s, i+r-m, {}^1/_2-s, -i, \\ n+1-s-i, i+r-n-s, {}^1/_2+r \end{matrix}\; 1\right],$$

$$\alpha = min\,(m-r,\ n-s),$$

$$\beta = min\,(m+n-r-s-2\varkappa,\ l-p).$$

5. Да разгледаме сега некои определени интеграли, во кои влегуваат придружените Legendre-ови сферни функции.

Најнапред да го пресметаме интегралот

$$\int_{-1}^{1} P_m^r(x)\, P_n^s(x)\, dx.$$

Од релацијата (3) имаме

$$\int_{-1}^{1} P_m^r(x)\, P_n^s(x)\, dx = 2^r \sum_{k=0}^{\lambda} \frac{A_{m-k}^{-r} A_{n-k}^{-s} A_{k,-r}}{A_{m+n-r-k}^{s}} \frac{(m+n-r-s-2k)\,!}{(m+n-r+s-2k)\,!}$$

$$\times \frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1}\; {}_4F_3 \left[\begin{matrix} r-s, k+r-m, {}^1/_2-s, -k, \\ n+1-k-s, k+r-m-n-s, {}^1/_2+r \end{matrix} \; 1 \right]$$

$$\times \int_{-1}^{1} (1-x^2)^{r/2}\, P_{m+n-r-2k}^{s}(x)\, dx.$$

За пресметување на последниот интеграл, зимаме во предвид дека е

$$P_n^s(x) = \frac{(n+s)\,!\, n!}{2^n} \sum_{k=0}^{n-s} (-1)^k \frac{(1-x)^{k+\frac{s}{2}} (1+x)^{n-\frac{s}{2}-k}}{(n-s-k)!\, k!\, (s+k)\,!\, (n-k)!}\,.$$

Тогај имаме

$$\int_{-1}^{1} (1-x^2)^{r/2} P_n^s(x)\, dx = \frac{(n+s)!\, n!}{2^n} \sum_{k=0}^{n-s} \frac{(-1)^k}{(n-s-k)!\, k!\, (s+k)!\, (n-k)!}$$

$$\times \int_{-1}^{1} (1-x)^{k+\frac{r+s}{2}} (1+x)^{n+\frac{r-s}{2}-k}\, dx.$$

Но бидејќи е

$$\int_{-1}^{1} (1-x)^{\frac{r+s}{2}+k} (1+x)^{n+\frac{r-s}{2}-k}\, dx =$$

$$= 2^{n+r+1} B\left(n+\frac{r-s}{2}-k+1, \frac{r+s}{2}+k+1\right),$$

ќе имаме

$$\int_{-1}^{1} (1-x^2)^{\frac{r}{2}} P_n^s(x)\, dx =$$

$$= 2^{r+1} (n+s)!\, n! \sum_{k=0}^{n-s} (-1)^k \frac{B\left(n+\frac{r-s}{2}-k+1, \frac{r+s}{2}+k+1\right)}{(n-s-k)!\, k!\, (s+k)!\, (n-k)!}.$$

Од друга страна лесно се наоѓа дека е

$$\sum_{k=0}^{n-s}(-1)^k\frac{B\left(n+\frac{r-s}{2}-k+1,\ \frac{r+s}{2}+k+1\right)}{(n-s-k)!\,k!\,(s+k)!\,(n-k)!}=$$

$$=\frac{\Gamma\left(\frac{r+s}{2}+1\right)\Gamma\left(\frac{r+s}{2}+\lambda+1\right)\left(\frac{s-r}{2}\right)_\lambda}{(r+s+2\lambda+2)\,(s+\lambda)!\,\lambda!\,n!},\ n-s=2\lambda$$

и добиваме

$$\int_{-1}^{1}(1-x)^{\frac{r}{2}}P_n^s(x)\,dx=0,\ n-s=2\lambda+1.$$

$$=\frac{2^{r+1}(n+s)!}{(s+\lambda)!\,\lambda!}\ \frac{\Gamma\left(\frac{r+s}{2}+1\right)\Gamma\left(\frac{r+s}{2}+\lambda+1\right)}{\Gamma(n+r+2)}\left(\frac{s+r}{2}\right)_\lambda,\ n-s=2\lambda$$

Во нашиот случај ќе имаме

$$\int_{-1}^{1}(1-x^2)^{\frac{r}{2}}P_{m+n-r-2k}^s(x)\,dx=\frac{2^{r+1}(m+n-r+s-2k)!}{(s+\nu-k)!\,(\nu-k)!}$$

$$\times\frac{\Gamma\left(\frac{r+s}{2}+\nu-k-1\right)\Gamma\left(\frac{r+s}{2}+1\right)}{\Gamma(m+n-2k+2)}\left(\frac{s-r}{2}\right)_{\nu-k},\ m+n-r-s=2k$$

$$=0,\ m+n-r-s=2r+1$$

Внесувајќи ја оваа вредност на интегралот во (5), добиваме

$$\int_{-1}^{1}P_m^r(x)\,P_n^s(x)\,dx=$$

$$=2^{2r+1}\,\Gamma\left(\frac{r+s}{2}+1\right)\sum_{k=0}^{\lambda}\frac{A_{m-k}^{-r}A_{n-k}^{-s}A_{k,-r}}{A_{m+n-r-k}^{s}}\ \frac{(2\nu-2k)!}{(s+\nu-k)!\,(\nu-k)!}$$

$$\times\frac{\Gamma\left(\frac{r+s}{2}+\nu-k+1\right)}{(m+n-2k+1)!}\ \frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+2}\left(\frac{s-r}{2}\right)_{\nu-k}$$

$$\times{}_4F_3\left[\begin{matrix}r-s,\ k+r-m,\ {}^1/_2-s,\ -k\\ n+1-k-s,\ k+r-m-n-s,\ {}^1/_2+r\end{matrix}\right],$$

и $$m+n-r-s=2\nu,$$

$$\int_{-1}^{1} P_m^r(x) P_n^s(x)\,dx = 0,\ m+n-r-s=2\nu+1.$$

6. Во случај кога е $r=s$, ја добиваме познатата формула

$$\int_{-1}^{1} P_m^r(x) P_n^r(x)\,dx = \begin{cases} 0, m \neq n \\ \dfrac{2}{2m+1}\,\dfrac{(m+r)!}{(m-r)!}, m=n. \end{cases}$$

Ако е $r=3, s=1$, го имаме следниот резултат [1]

$$\int_{-1}^{1} P_m^3(x) P_n^1(x)\,dx = \begin{cases} 8n(n+1), m-n=2, \\ -\dfrac{2n+1}{2}\dbinom{n+1}{4}4!, m=n, \\ 0, \text{останати вредности на } m \text{ и } n. \end{cases}$$

Од релацијата (3) може да се добие и формулата

$$\int_{-1}^{1} (1-x^2)^{-r/2} P_m^r(x) P_n^s(x)\,dx$$

$$= 2^{r+1} \sum_{k=0}^{\lambda} \frac{A_{m-k}^{-r} A_{n-k}^{-s} A_{k,-r}}{A_{m+n-r-k}^{s}} \frac{(2\nu-2k)!}{(s+\nu-k)!\,(\nu-k)!}$$

$$\times \left(\frac{s}{2}\right)_{\nu-k} \frac{\Gamma\left(\frac{s}{2}+1\right)\Gamma\left(\frac{s}{2}+\nu-k+1\right)}{\Gamma(m+n-r-2k+2)} \frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1}$$

$$\times {}_4F_3\left[\begin{matrix} r-s, k+r-m, {}^1/_2-s, -k, \\ n+1-s-k, k+r-m-n-s, {}^1/_2+r, \end{matrix} 1\right].$$

Лесно се добива сега и формулата

$$\int_{-1}^{1} (1-x^2)^{-r/2} P_m^r(x) P_n(x)\,dx =$$

$$= 2\binom{k+r-1}{r-1}\frac{(m+n+r-1)!!}{(m+n-r+1)!!}, m-r-n=2k\geqslant 0$$

$$= 0,\ m-r-n<0 \text{ или } m-n-r=2\kappa+1.$$

$$(1 \leqslant r \leqslant m).$$

како и

$$\int_{-1}^{1} P_m(x)\, P_n^s(x)\, dx = 2 \sum_{k=0}^{\lambda} \frac{A_{m-k} A_k A_{n-k}^{-s}}{A_{m+n-k}^{s}} \frac{(2\nu-2k)!}{(s+\nu-k)!\,(\nu-k)!} \left(\frac{s}{2}\right)_{\nu-k}$$

$$\times \frac{2m+2n-4k+1}{2m+2n-2k+1} \frac{\Gamma\left(\frac{s}{2}+1\right)\Gamma\left(\frac{s}{2}+\nu-k+1\right)}{\Gamma(m+n-2k+2)}$$

$${}_4F_3\left[\begin{matrix} -s, k-m, {}^1/_2-s, -k, \\ n+1-k, k-m-n-s, {}^1/_2 \end{matrix}\; 1\right]$$

$$(m+n-r=2\nu).$$

7. На сличен начин може да се пресмета интегралот во којшто влегува производ од три придружени Legendre-ови сферни функции. За тоа цел поаѓаме од релацијата (4) и наоѓаме

$$\int_{-1}^{1} P_l^p(x)\, P_m^r(x)\, P_n^s(x)\, dx = 2^{2r+2s+1}\, \Gamma\left(\frac{p+r+s}{2}+1\right)$$

$$\times \sum_{\varkappa=0}^{\alpha} \sum_{\nu=0}^{\beta} B_{m,n,\varkappa}^{r,s} B_{m+n-r-2\varkappa,\, l,\, \nu}^{s,p} \frac{(2\lambda+2p-2\varkappa-2\nu)!}{(l+m+n-2\varkappa-2\nu+1)}$$

$$\times \frac{\Gamma\left(\frac{p+r+s}{2}+\lambda-\varkappa-\nu+1\right)}{(\lambda-\varkappa-\nu)!\,(p+\lambda-\varkappa-\nu)!} \left(\frac{p-r-s}{2}\right)_{\lambda-\varkappa-\nu},$$

$$l+m+n-p-r-s=2\lambda.$$

Од (3) може да се определи и интегралота формула [1]*

$$\int_{-1}^{1} (1-x^2)^{-r/2} P_l^r(x)\, P_m^r(x)\, P_n^r(x)\, dx =$$

$$= \frac{2^r\, \Gamma(m-k+{}^1/_2)\, \Gamma(n-k+{}^1/_2)\, \Gamma(r+k+{}^1/_2)\,(m+n-k)!}{k!\,(m-r-k)!\,(n-r-k)!\,\Gamma({}^1/_2)\,\Gamma(l+k+{}^3/_2)\,\Gamma(r+{}^1/_2)},$$

$$m+n-r-l=2\kappa.$$

8. Да го пресметаме сега интегралот

$$\int_{0}^{1} P_m^r(x)\, P_n^r(x)\, dx.$$

Знаеме дека е [4]

* Формулата е дадена од Dougall, [2] и содржи штампарска грешка.

$$\int_0^1 (1-x^2)^{r/2} P_n^r(x)\,dx = \frac{\left(\frac{r-n}{2}+1\right)_n (n+r)!\;\Gamma(^1/_2)}{2^{r+1}(n-r)!\;\Gamma\left(\frac{n+r}{2}+1\right)\Gamma\left(\frac{n}{2}+\frac{r}{2}+\frac{3}{2}\right)}\,.$$

Во тој случај, зимајќи во предвид (3), ќе имаме

$$\int_0^1 P_m^r(x)\,P_n^r(x)\,dx =$$

$$= 2^{-1}\sum_{k=0}^{\lambda} \frac{A_{m-k}^{-r} A_{n-k}^{-r} A_{k,-r}}{A_{m+n-r-k}^{r}}\;\frac{\left(k+1-\frac{m+n}{2}\right)_{m+n-r-2k}}{\Gamma\left(\frac{m+n}{2}-k+1\right)\Gamma\left(\frac{m+n+3}{2}+k\right)}$$

$$\times\frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1}\,\Gamma(^1/_2),$$

$$\lambda = min\,(m-r,\; n-r).$$

По истиот начин, зимајќи во предвид дека е

$$\int_0^1 x^s(1-x^2)^{r/2} P_n^r(x)\,dx = \frac{(-1)^r\,\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)\Gamma\left(1+\frac{s}{2}\right)(r+n)!}{2^{r+1}(n-r)!\;\Gamma\left(1+\frac{s}{2}+\frac{r}{2}-\frac{n}{2}\right)\Gamma\left(\frac{3}{2}+\frac{s}{2}+\frac{n}{2}+\frac{r}{2}\right)}$$

наоѓаме

$$\int_0^1 x^s P_m^r(x)\,P_n^r(x)\,dx =$$

$$= \frac{(-1)^r\,\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)\Gamma\left(1+\frac{s}{2}\right)}{2}\sum_{k=0}^{\lambda}\frac{A_{m-k}^{-r} A_{k,-r} A_{n-k}^{-r}}{A_{m+n-k-r}^{r}}\;\frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1}$$

$$\times\frac{1}{\Gamma\left(1+r+k+\frac{s}{2}-\frac{m+n}{2}\right)\Gamma\left(\frac{3}{2}+\frac{s}{2}+\frac{m+n}{2}+k\right)}\,.$$

$$\lambda = min\,(m-r,\; n-r).$$

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[2] Bailey, W. N (1935), *On the Product of two associated Legendre Functions*, The Quaterly Journal of Mathematics, Oxford series, vol 11, № 41 (1940). pp. 30—36.

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Summary

ON ASSOCIATED LEGENDRE FUNCTIONS

1. The object of the present note is to obtain a generalisation of Bailey's formula (1) concerning of the product of two associated Legendre functions. We deduce also, some integral formulas involving these functions.

We profiti by the Ferrers definition of the associated Legendre functions (2)

$$P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m P_n(x)}{dx^m},$$

$$m = 0,1,2,\dots;\ n \text{ being unrestricted},$$

or, in hyrergeometric notation,

$$P_n^m(x) = \frac{(n+m)!}{2^m m!(n-m)!}(1-x^2)^{m/2} F\left(m-n, m+n+1, m+1, \frac{1-x}{2}\right).$$

2. Recently we have established the following result [3].

If m, n are positive integers, and $r < m$, $s < n$, then

$$\text{(1)}\quad F\left(r-m, m+r+1, r+1, \frac{1-x}{2}\right) F\left(s-n, n+s+1, s+1, \frac{1-x}{2}\right)$$

$$= \sum_{k=0} a_k F\left(r+s-m-n+2k, m+n-r+s-2k+1, s+1, \frac{1-x}{2}\right),$$

with

$$\text{(2)}\qquad a_k = 2^{2r} \frac{(m-r)!\,(n-s)!\,r!}{(m+r)!\,(n+s)!} \frac{A_{m-k} A_{n-k} A_{k,-r}}{A^s_{m+n-r-k}}$$

$$\times \frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1}\, {}_4F_3\left[\begin{matrix} r-s, k+r-m, {}^1/_2-s, -k, \\ n-s-k+1, r+k-m-n-s, {}^1/_2+r \end{matrix}\ 1\right],$$

where

$$A_m = \frac{\left(\frac{1}{2}\right)_r}{(m+r)!},\quad A_{k,r} = \frac{\left(\frac{1}{2}+r\right)_k}{k!},$$

$${}_4F_3\left[\begin{matrix}\alpha, \beta, \gamma, \delta \\ a, b, c\end{matrix}\ x\right] = \sum_{i=0}^{\infty} \frac{(\alpha)_i (\beta)_i (\gamma)_i (\delta)_i}{(a)_i (b)_i (c)_i} \frac{x^i}{i!},$$

$$(\alpha)_i = \alpha(\alpha+1)\dots(\alpha+i-1),\ (\alpha)_0 = 1,$$

$$\lambda = min\,(m-r, n-s).$$

From (1) and (2) we have

(3) $$(1-x^2)^{-r/2}\, P_m^r(x) P_n^s(x)$$

$$= 2^r \sum_{k=0}^{\lambda} \frac{A_{m-k}^{-r} A_{n-k}^{-s} A_{k,-r}}{A_{m+n-r-k}^{s}} \frac{(m+n-r-s-2k)!}{(m+n-r+s-2k)!}$$

$$\times \frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1} {}_4F_3\left[\begin{matrix} r-s, k+r-m, 1/2-s, -k, \\ n+1-k-s, k+r-m-n-s, 1/2+r \end{matrix} 1\right] P_{m+n-r-2k}^{s}(x).$$

3. Now for the special cases of r and s we obtain

1° If $r = s$, the Bailey's formula

$$(1-x^2)^{-r/2}\, P_m^r(x) P_n^r(x) = 2^r \sum_{k=0}^{\lambda} \frac{A_{m-k}^{-r} A_{n-k}^{-s} A_{k,-r}}{A_{m+n-r-k}^{s}} \frac{(m+n-2r-2k)}{(m+n-2k)}$$

$$\times \frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1} P_{m+n-r-2k}^{r}(x),$$

$$\lambda = min\,(m-r,\ n-s).$$

2°. If $r = s = 0$, the Neumann — Adams formula

$$Pm(x)\, Pn(x) = \sum_{k=0}^{min\,(m,n)} \frac{A_{m-k} A_{n-k} A_k}{A_{m+n-k}} \frac{2m+2n-4k+1}{2m+2n-2k+1} P_{m+n-2k}(x).$$

We easily also find

$$Pm(x) P_n^s(x) = \sum_{k=0}^{min\,(m,\,n-s)} \frac{A_{m-k} A_{n-k}^{-s} A_k}{A_{m+n-k}^{s}} \frac{(m+n-s-2k)!}{(m+n+s-2k)!} \frac{2m+2n-4k+1}{2m+2n-2k+1}$$

$$\times {}_4F_3\left[\begin{matrix} -s, k-m, 1/2-s, -k, \\ n+1-s, -k, k-m-n-s, 1/2 \end{matrix} 1\right] P_{m+n-2k}^{s}(x),$$

and

$$Pm(x) P_n^s(x) = 2^s \sum_{k=0} \frac{A_{n-k}^{-s} A_{m-k} A_{k,-s}}{A_{m+n-s-k}} \frac{2m+2n-2s-4k+1}{2m+2n-2s-2k+1}$$

$$\times {}_4F_3\left[\begin{matrix} s, k+s-n, 1/2, -k, \\ m-k+1, k+s-m-n, 1/2+s \end{matrix} 1\right] (1-x^2)^{s/2}\, P_{m+n-s-2k}(x).$$

4. A further generalisation of the result (3) is the following

$$P_l^p(x)\, P_m^r(x) P_n^s(x) = 2^{r+s} \sum_{k=0}^{\lambda} \sum_{v=0}^{m} B_{m,n,k}^{r,s}\, B_{m+n-s-2k,\,l,\,v}^{s,p}$$

$$\times (1-x^2)^{\frac{r+s}{2}} P^{p}_{l+m+n-r-s-2k-2\nu}(x),$$

where

$$B^{r,s}_{m,n,i} = \frac{A^{-r}_{m-i} A^{-s}_{n-i} A_{i,-r}}{A^{s}_{m+n-r-k}} \frac{(m+n-r-s-2i)!}{(m+n-r+s-2i)!}$$

$$\times \frac{2m+2n-2r-4i+1}{2m+2n-2r-2i+1} {}_4F_3 \left[\begin{matrix} r-s, i+r-m, {}^1/_2-s, -i, \\ n+1-s-i, i+r-m-n-s, {}^1/_2+r \end{matrix} \; 1 \right],$$

$$\lambda = min\ (m-r, n-s),$$

$$\mu = min\ (l-p, m+n-r-s-2\varkappa).$$

5. With the help the above result, we now evaluate the integral of the product of two associated Legendre functions of different degree and different order.

Since

$$\int_{-1}^{1} (1-x)^{\frac{r}{2}} P^{s}_{n}(x)\, dx = 0 \quad, n-s = 2\lambda+1$$

$$= \frac{2^{r+1}(n+s)!\ \Gamma\left(\frac{r+s}{2}+1\right)\ \Gamma\left(\frac{r+s}{2}+\lambda+1\right)}{(s+\lambda)!\ \lambda!\ \ \Gamma(n+r+2)} \left(\frac{s-r}{2}\right)_{\lambda}, n-s = 2\lambda$$

we have

$$\int_{-1}^{1} P^{r}_{m}(x)\, P^{s}_{n}(x)\, dx = 2^{2r+1}\, \Gamma\left(\frac{r+s}{2}+1\right) \sum_{k=0}^{\lambda} \frac{A^{-r}_{m-k} A_{k,-r} A^{-s}_{n-k}}{A^{s}_{m+n-r-k}} \frac{(2\nu-2k)!}{(s+\nu-k)!(\nu-k)}$$

$$\times \frac{\Gamma\left(\frac{r+s}{2}+\nu-k+1\right)}{\Gamma(m+u-2k+2)} \frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1} \left(\frac{s-r}{2}\right)_{\nu-k}$$

$$\times {}_4F_3 \left[\begin{matrix} r-s, k+r-m, {}^1/_2-s, -k, \\ n-s+1-k, k+r-m-k-s, {}^1/_2+r, \end{matrix} \; 1 \right]$$

$$m+n-r-s = 2\nu,$$

and

$$\int_{-1}^{1} P^{r}_{m}(x)\, P^{s}_{n}(x)\, dx = 0, \ m+n-s-r = 2n+1.$$

When $r = s$, we get the well-known result

$$\int_{-1}^{1} P^{r}_{m}(x)\, P^{r}_{n}(x) = 0, \quad m \neq n$$

$$= \frac{2}{2n+1} \frac{(n+r)!}{(n-r)!}, m = n.$$

If $s = 0$, we have

$$\int_{-1}^{1} (1-x^2)^{-r/2} P_n(x) P_m^r(x)\, dx = 2\binom{k+r-1}{r-1}\frac{(m+n+r-1)!!}{(m+n-r+1)!!},\ m-n-r=2k \geqslant 0,$$

$$= o,\ m-r-n=0 \text{ or } m-n-r = 2k+1$$

$$(1 \leqq r \leqslant m).$$

Similarly

$$\int_{0}^{1} P_m^r(x) P_n^r(x)\, dx = \frac{\sqrt{\pi}}{2}\sum_{k=0}^{\lambda} \frac{A_{m-k} A_{k,-r} A_{n-k}}{A^r_{m+n-r-k}} \frac{\left(k+1-\frac{m+n}{2}\right)_{m+n-r-2k}}{\Gamma\left(\frac{m+n}{2}-k+1\right)\Gamma\left(\frac{m+n+3}{2}+k\right)}$$

$$\times \frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1},$$

$$\lambda = min\ (m-r,\ n-r).$$

6. Following the above method, we obtain

$$\int_{-1}^{1} P_l^r(x) P_m^r(x) P_n^s(x)\, dx =$$

$$= 2^{2r+2s+1}\ \Gamma\left(\frac{p+r+s}{2}+1\right) \sum_{k=0} \sum_{\nu=0} B^{r,s}_{m,n,k}\ B^{s,p}_{m+n-r-2k,\,l,\,\nu}$$

$$\times \frac{(2\lambda-2p-2k-2\nu)!}{(l+m+n-2k-2\nu+1)!} \frac{\Gamma\left(\frac{p+r+s}{2}+\lambda-k-\nu+1\right)}{(\lambda-k-\nu)!\,(p+\lambda-k-\nu)!} \left(\frac{p-r-s}{2}\right)_{\lambda-k-\nu}$$

$$l+m+n-p-r-s = 2\lambda.$$

We further obtain a very simple proof of Dougall's formula

$$\int_{-1}^{1} (1-x^2)^{-r/2}\ P_l^r(x)\ P_m^r(x) P_n^r(x)\, dx$$

$$= \frac{2^r\,\Gamma(m-k+1/2)\ \Gamma(n-k+1/2)\ \Gamma(r+k+1/2)\ (m+n-k)!}{k!\,(m-r-k)!\,(n-r-k)!\ \Gamma(1/2)\ \Gamma(l+k+3/2)\ \Gamma(r+1/2)},$$

$$m+n-r-l = 2k.$$