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SOME CHARACTERIZATIONS OF 2-INNER PRODUCT

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Abstract. The characterization of 2-inner product is an issue which is the focus of interest of many mathematicians. In this paper, several equivalent characterizations of 2-inner product, that are consequences of Theorem 2 ([12]) are discussed. Thus, the equivalence of generalizations of the Jordan and von Neumann ([7]) and also Frechet ([8]) classical results, are proven. Furthermore, the characterization of Hlawka, the characterization of D. S. Marinescu, M. Monea, M. Opincariu and M. Stroe ([4]) are proven as well.

1. INTRODUCTION

Let L be a real vector space with dimension greater than 1 and $||\cdot,\cdot||$ be a real function on $L\times L$ such that:

a) $||x,y||\geq 0$, for all $x,y\in L$ and $||x,y||=0$ if and only if the set $\{x,y\}$ is linearly dependent,

b) $||x,y||=||y,x||$, for all $x,y\in L$,

c) $||\alpha x,y||=|\alpha|\cdot||x,y||$, for all $x,y\in L$ and for each $\alpha\in\mathbf{R}$, and

d) $||x+y,z||\leq||x,z||+||y,z||$, for all $x,y,z\in L$.

The function $||\cdot,\cdot||$ is said to be 2-norm of L, and $(L,||\cdot,\cdot||)$ is said to be vector 2-normed space ([11]). The inequality in the axiom d) is said to be parallelepiped inequality.

Let $n>1$ be a positive integer, L be a real vector space, $\dim L\geq n$ and $(\cdot,\cdot|\cdot)$ be a real function over $L\times L\times L$ such that:

i) $(x,x|y)\geq 0$, for all $x,y\in L$ and $(x,x|y)=0$ if and only if x and y are linearly dependent,

ii) $(x,y|z)=(y,x|z)$, for all $x,y,z\in L$,

iii) $(x,x|y)=(y,y|x)$, for all $x,y\in L$,

iv) $(\alpha x,y|z)=\alpha(x,y|z)$, for all $x,y,z\in L$.and for each $\alpha\in\mathbf{R}$, and

v) $(x+x_1,y|z)=(x,y|z)+(x_1,y|z)$, for all $x_1,x,y,z\in L$.

The function $(\cdot,\cdot|\cdot)$is said to be 2-inner product, and $(L,(\cdot,\cdot|\cdot))$ is said to be 2-pre-Hilbert space ([4]).

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The concepts of 2-norm and 2-inner product are two dimensional analogies to the concepts of a norm and an inner product. R. Ehret proved ([9]) that if $(L, (\cdot, \cdot|\cdot))$ is a 2-pre-Hilbert space, then

$$||x, y|| = (x, x|y)^{1/2} \tag{1}$$

for all $x, y \in L$, defines a 2-norm.

So, we get vector 2-normed space $(L, ||\cdot, \cdot||)$ and moreover, for all $x, y, z \in L$ the following equalities are satisfied:

$$(x, y|z) = \frac{||x+y,z||^2 - ||x-y,z||^2}{4} \tag{2}$$

$$||x + y, z||^2 + ||x - y, z||^2 = 2(||x, z||^2 + ||y, z||^2) \tag{3}$$

The equality (3) is actually analogous to the parallelogram equality and it is called parallelepiped equality. Further, 2-normed space L is 2-pre-Hilbert space if and only if for all $x, y, z \in L$ the equality (3) holds true.

The following theorem gives one other elementary characterization of 2-inner product, i.e. we will prove the equivalence of generalizations of the classical results of Jordan and von Neumann ([7]) and Frechet ([8]).

Theorem 1. Let $(L, ||\cdot, \cdot||)$ be a real 2-normed space. Then, L is a 2-pre-Hilbert space if and only if

$$\begin{aligned} &||x + y + z, u||^2 + ||x, u||^2 + ||y, u||^2 + ||z, u||^2 = \\ &\quad = ||x + y, u||^2 + ||y + z, u||^2 + ||z + x, u||^2 \end{aligned} \tag{4}$$

for all $x, y, z, u \in L$.

Proof. Let the equality (4) hold true, for all $x, y, z, u \in L$. For $z = -y$, and for each $x, y, u \in L$ it is true that

$$||x, u||^2 + ||x, u||^2 + ||y, u||^2 + ||-y, u||^2 = ||x+y, u||^2 + ||0, u||^2 + ||x-y, u||^2,$$

i.e. the equality (3) holds true, therefore, L is a 2-pre-Hilbert space. Conversely, let $(L, ||\cdot, \cdot||)$ be a 2-pre-Hilbert space. Then the equation of the parallelogram implies that for all $x, y, z, u \in L$ the following holds true

$$\begin{aligned} &||x + y + z, u||^2 + ||x, u||^2 + ||y, u||^2 + ||z, u||^2 = \\ &\qquad = \tfrac{1}{2}||2x + y + z, u||^2 + \tfrac{1}{2}||y + z, u||^2 + \tfrac{1}{2}||y + z, u||^2 + \tfrac{1}{2}||y - z, u||^2 \\ &\qquad = \tfrac{1}{2}||2x + y + z, u||^2 + \tfrac{1}{2}||y - z, u||^2 + ||y + z, u||^2 \\ &\qquad = \tfrac{1}{2}||(x + y) + (x + z), u||^2 + ||(x + y) - (x + z), u||^2 + ||y + z, u||^2 \\ &\qquad = ||x + y, u||^2 + ||y + z, u||^2 + ||z + x, u||^2, \end{aligned}$$

i.e. the equality (4) holds true.

2. CHARACTERIZATION OF 2-PRE-HILBERT SPACE

The problem of characterization of 2-pre-Hilbert spaces, i.e. the necessary and sufficient conditions, the 2-normed spaces to be treated as 2-pre-Hilbert space is of particular interest while studying the 2-normed spaces. Thus, in [5] characterization of 2-pre-Hilbert space is given by using Euler-Lagrange type of equality, in [7] is given characterization by using strictly convex norm with modulus c, and in [9] are given characterizations by using Mercer inequality for 2-normed space and its equivalent inequality. Furthermore, in [13] the following theorem is proven.

Theorem 2. ([12]). Let $(L,||\cdot,\cdot||)$ be a real 2-normed space. L is a 2-pre-Hilbert space if and only if the following condition is satisfied: If $n\geq 3$, $x_1,x_2,...,x_n,z\in L$ and $a_1,a_2,...,a_n$ are real numbers such that $\sum\limits_{i=1}^{n} a_i=0$, then

$$||\sum_{i=1}^{n} a_i x_i, z||^2 = -\sum_{1\leq i<j\leq n} a_i a_j ||x_i-x_j,z||^2 \tag{5}$$

.

In the next consequence, by applying theorems 1 and 2 we will prove the following generalization of Hlawkas result.

Corollary 1. Let $(L,||\cdot,\cdot||)$ be a real 2-normed space. L is a 2-pre-Hilbert space if and only if for each $n\geq 2$ and for each $x_1,x_2,...,x_n,z\in L$ the following equality holds true

$$||\sum_{i=1}^{n} x_i, z||^2 + (n-2)\sum_{i=1}^{n}||x_i,z||^2 = \sum_{1\leq i<j\leq n}||x_i+x_j,z||^2. \tag{6}$$

Proof. Let for each $n\geq 2$ and for all $x_1,x_2,...,x_n,z\in L$ the equality (6) holds true. For $n=3$ and $x_1=x,\ x_2=y,\ x_3=u$ the equality (6) is transformed as the following:

$$||x+y+u,z||^2+||x,z||^2+||y,z||^2+||u,z||^2=$$

$$=||x+y,z||^2+||y+u,z||^2+||u+x,z||^2,$$

$x,y,u,z\in L$

by applying Theorem 1 we get that L is a 2-pre-Hilbert space.

Conversely, let L be a 2-pre-Hilbert space, $n\geq 2$ and $x_1,x_2,...,x_n,z$ are arbitrary vectors from L. If we get that $a_{n+1}=-n,\ a_i=1,\ i=1,2,...,n$ and $x_{n+1}=0$ then, by applying the above theorem and the parallelepiped equality the following equality holds true

$$||\sum_{i=1}^{n} x_i, z||^2 = ||\sum_{i=1}^{n} x_i - nx_{n+1}, z||^2 =$$
$$= n\sum_{i=1}^{n} ||x_i - x_{n+1}, z||^2 - \sum_{1\le i<j\le n} ||x_i - x_j, z||^2$$
$$= n\sum_{i=1}^{n} ||x_i, z||^2 - 2\sum_{1\le i<j\le n} (||x_i, z||^2 + ||x_j, z||^2) + \sum_{1\le i<j\le n} ||x_i + x_j, z||^2)$$
$$= n\sum_{i=1}^{n} ||x_i, z||^2 - 2(n-1)\sum_{i=1}^{n} ||x_i, z||^2 + \sum_{1\le i<j\le n} ||x_i + x_j, z||^2),$$

which is equivalent to equality (6).

In the following considerations will be proven several consequences of Theorem 2, which are actually generalizations of the results given by D. S. Marinescu, M. Monea, M. Opincariu and M. Stroe ([4]).

Corollary 2. Let $(L, ||\cdot,\cdot||)$ be a real 2-normed space. L is a 2-pre-Hilbert space if and only if for each $n \ge 2$, for all $a_1, a_2, ..., a_n \in \mathbf{R}$ and for all $x_1, x_2, ..., x_n, z \in L$ the following equality holds true

$$||\sum_{i=1}^{n} a_i x_i, z||^2 =$$
$$= (a_1 + a_2 + ... + a_n)\sum_{i=1}^{n} a_i ||x_i, z||^2 - \sum_{1\le i<j\le n} a_i a_j ||x_i - x_j, z||^2. \qquad (7)$$

Proof. For each $n \ge 2$, and for all $a_1, a_2, ..., a_n \in \mathbf{R}$ and $x_1, x_2, ..., x_n, z \in L$, let the equality (7) hold true. For $n = 2$, $a_1 = a_2 = 1$, $x_1 = x$, $x_2 = y$ the equality (7) is transformed as the following

$$||x + y, z||^2 = 2(||x, z||^2 + ||y, z||^2) - ||x - y, z||^2, x, y, z \in L$$

i.e. the equality (3) holds true. The latter means that L is a 2-pre-Hilbert space.

Conversely, let L be a 2-pre-Hilbert space, $n \ge 2$, $a_1, a_2, ..., a_n$ be arbitrary real numbers and $x_1, x_2, ..., x_n, z$ arbitrary vectors on L.

Let $a_{n+1} = -(a_1 + a_2 + ... + a_n)$ and $x_{n+1} = 0$. Since Theorem 2 we get the following

$$||\sum_{i=1}^{n} a_i x_i, z||^2 = ||\sum_{i=1}^{n+1} a_i x_i, z||^2 = -\sum_{1\le i<j\le n+1} a_i a_j ||x_i - x_j, z||^2$$
$$= -\sum_{i=1}^{n} a_i a_{n+1} ||x_i - x_{n+1}, z||^2 - \sum_{1\le i<j\le n} a_i a_j ||x_i - x_j, z||^2$$
$$= -(a_1 + a_2 + ... + a_n)\sum_{i=1}^{n} a_i ||x_i, z||^2 - \sum_{1\le i<j\le n} a_i a_j ||x_i - x_j, z||^2,$$

The latter means that the equality (7) holds true.

Corollary 3. Let $(L, ||\cdot,\cdot||)$ be a real 2-normed space. L is a 2-pre-Hilbert space if and only if for each $n \geq 2$, for all $a_1, a_2, ..., a_n \in \mathbf{R}$ such that $\sum\limits_{i=1}^{n} a_i = 1$ and for all $x_1, x_2, ..., x_n, z \in L$ the following equality holds true

$$||\sum_{i=1}^{n} a_i x_i, z||^2 = \sum_{i=1}^{n} a_i ||x_i, z||^2 - \sum_{1 \leq i < j \leq n} a_i a_j ||x_i - x_j, z||^2. \tag{8}$$

Proof. Let for each $n \geq 2$, for all $a_1, a_2, ..., a_n \in \mathbf{R}$ so that $\sum\limits_{i=1}^{n} a_i = 1$ and $x_1, x_2, ..., x_n,\, z \in L$, the equality (8) hold true. For $n = 2$, $a_1 = a_2 = \frac{1}{2}$, $x_1 = x,\ x_2 = y$ the equality (8) is transformed as following

$$||\tfrac{1}{2}x + \tfrac{1}{2}y, z||^2 = \tfrac{1}{2}||x, z||^2 + \tfrac{1}{2}||y, z||^2 - \tfrac{1}{4}||x + y, z||^2$$

for all $x, y, z \in L$. The latter is equivalent to the equation (3). Therefore L is 2-pre-Hilbert space.

Conversely, let L be a 2-pre-Hilbert space and $n \geq 2$, $a_1, a_2, ..., a_n$ are any real numbers so that $\sum\limits_{i=1}^{n} a_i = 1$ and $x_1, x_2, ..., x_n, z$ are any vectors on L. Let $a_{n+1} = -1$ and $x_{n+1} = 0$. Theorem 2 implies the following

$$\begin{aligned}
||\sum_{i=1}^{n} a_i x_i, z||^2 = ||\sum_{i=1}^{n+1} a_i x_i, z||^2 &= -\sum_{1 \leq i < j \leq n+1} a_i a_j ||x_i - x_j, z||^2 \\
&= -\sum_{i=1}^{n} a_i a_{n+1} ||x_i - x_{n+1}, z||^2 - \sum_{1 \leq i < j \leq n} a_i a_j ||x_i - x_j, z||^2 \\
&= \sum_{i=1}^{n} a_i ||x_i, z||^2 - \sum_{1 \leq i < j \leq n} a_i a_j ||x_i - x_j, z||^2,
\end{aligned}$$

The latter means that (8) holds true.

Corollary 4. Let $(L, ||\cdot,\cdot||)$ be a real 2-normed space. Then L is a 2-pre-Hilbert space if and only if for all $n \geq 2$, for each $a_1, a_2, ..., a_n \in \mathbf{R}\backslash\{0\}$ so that $\sum\limits_{i=1}^{n} a_i \neq 0$ and for all $x_1, x_2, ..., x_n, z \in L$ the following equality holds true

$$\sum_{i=1}^{n} \tfrac{||x_i, z||^2}{a_i} - \tfrac{1}{\sum\limits_{i=1}^{n} a_i} ||\sum_{i=1}^{n} x_i, z||^2 = \tfrac{1}{\sum\limits_{i=1}^{n} a_i} \sum_{1 \leq i < j \leq n} \tfrac{||a_j x_i - a_i x_j, z||^2}{a_i a_j}. \tag{9}$$

Proof. Let for each $n \geq 2$, for all $a_1, a_2, ..., a_n \in \mathbf{R}\backslash\{0\}$ so that $\sum\limits_{i=1}^{n} a_i \neq 0$ and for all $x_1, x_2, ..., x_n, z \in L$ the equality (9) holds true. For $n = 2$, $a_1 = a_2 = 1$, $x_1 = x,\ x_2 = y$ the equality (9) is the following

$$||x, z||^2 + ||y, z||^2 - \tfrac{1}{2}||x + y, z||^2 = \tfrac{1}{2}||x - y, z||^2,$$

for all$x, y, z \in L$. The latter is equivalent to (3). Thus, $L$is 2-pre-Hilbert space.

Conversely, let L be a 2-pre-Hilbert space and $n \geq 2$, $a_1, a_2, ..., a_n \in \mathbf{R}\backslash\{0\}$ are arbitrary real numbers so that $\sum\limits_{i=1}^{n} a_i \neq 0$ holds true and $x_1, x_2, ...,$ x_n, z are arbitrary vectors on L. Let $a_{n+1} = -1$ and $x_{n+1} = 0$.

Theorem 2 implies the following

$$\begin{aligned} ||\sum_{i=1}^{n} a_i x_i, z||^2 = ||\sum_{i=1}^{n+1} a_i x_i, z||^2 &= -\sum_{1\leq i<j\leq n+1} a_i a_j ||x_i - x_j, z||^2 \\ &= -\sum_{i=1}^{n} a_i a_{n+1} ||x_i - x_{n+1}, z||^2 - \sum_{1\leq i<j\leq n} a_i a_j ||x_i - x_j, z||^2 \\ &= \sum_{i=1}^{n} a_i ||x_i, z||^2 - \sum_{1\leq i<j\leq n} a_i a_j ||x_i - x_j, z||^2, \end{aligned}$$

The latter means that (9) holds true.

Corollary 5. Let $(L, ||\cdot,\cdot||)$ be a real 2-normed space. L is a 2-pre-Hilbert space if and only if for each $n \geq 2$ and for all $x_1, x_2, ..., x_n, z \in L$ the following holds true

$$\tfrac{1}{n}||\sum_{i=1}^{n} x_i, z||^2 = \sum_{i=1}^{n} ||x_i, z||^2 - \tfrac{1}{n}\sum_{1\leq i<j\leq n} ||x_i - x_j, z||^2. \tag{10}$$

.

Proof. Let (10) hold true for each $n \geq 2$ and for all $x_1, x_2, ..., x_n, z \in L$. For $x_1 = x,\ x_2 = y$ (10) is transformed as the following

$$\tfrac{1}{2}||x + y, z||^2 = ||x, z||^2 + ||y, z||^2 - \tfrac{1}{2}||x - y, z||^2,$$

for all $x, y, z \in L$. The above equality is equivalent to (3). Thus, $L$is 2-pre-Hilbert space. Conversely, let L be a 2-pre-Hilbert space and $n \geq 2$ and $x_1, x_2, ..., x_n, z$ are arbitrary vectors on L. Let $a_{n+1} = -1,\ a_i = n,\ i = 1, 2, ..., n$ and $x_{n+1} = 0$. Theorem 2 implies the following

$$\begin{aligned}||\sum_{i=1}^{n}\tfrac{1}{n}x_i,z||^2 &= ||\sum_{i=1}^{n}\tfrac{1}{n}x_i-x_{n+1},z||^2 = \\ &=\tfrac{1}{n}\sum_{i=1}^{n}||x_i-x_{n+1},z||^2-\sum_{1\le i<j\le n}\tfrac{1}{n^2}||x_i-x_j,z||^2 \\ &=\tfrac{1}{n}\sum_{i=1}^{n}||x_i,z||^2-\tfrac{1}{n^2}\sum_{1\le i<j\le n}||x_i-x_j,z||^2,\end{aligned}$$

The latter is equivalent to (10).

CONFLICT OF INTEREST
No conflict of interest was declared by the author.

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