Математички Билтен
42(LXVIII) No. 1
2018(21-26)
Скопје, Македонија

ISSN 0351-336X (print)
ISSN 1857-9914 (online)
UDC: 517.521:515.124

CONVERGENCE OF SEQUENCES IN $(3, j, \rho)$-N-METRIZABLE SPACES, $j \in \{1, 2\}$

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Abstract. In this paper we consider $(3, j, \rho)$-N-metrizable spaces,$j \in \{1, 2\}$, in terms of convergence of sequences. We define four types of convergence in these spaces. We give examples which show that in general these four types of convergence are not equivalent in $(3, 1, \rho)$-N-metrizable spaces. We show that in $(3, 2)$-N-metrizable spaces these four types of convergence are equivalent.

1. INTRODUCTION

The geometric properties, their axiomatic classification and the generalization of metric spaces have been considered in [11]. Later, several notions of generalized metrics have been introduced. Some of them are: o-metric, symmetric, pseudo-o-metric in [13] , 2-metric in [8] and [9], G-metric in [12]. The notion of (n, m, ρ)-metric is introduced in [3]. Connections between some of the topologies induced by a $(3, 1, \rho)$-metric d and topologies induced by a pseudo-o-metric, o-metric and symmetric are given in [4]. For a given $(3, j, \rho)$-metric d on a set M, $j \in \{1, 2\}$, seven topologies $\tau(G, d)$, $\tau(H, d)$, $\tau(D, d)$, $\tau(N, d)$,$\tau(W, d)$,$\tau(S, d)$ and $\tau(K, d)$ on M, induced by d, are defined in [1], and several properties of these topologies are shown in [1], [2], [5], [6] and [7].

Here we consider only $(3, j, \rho)$- N-metrizable spaces, $j \in \{1, 2\}$, in terms of convergence of sequences and investigate some properties about them. We use notations as in [1].

First we will define some of these notions.

Let M be a nonempty set and let $M^{(3)} = M^3/\alpha$, where α is the equivalence relation on M^3 defined by:

$(x, y, z)\alpha(u, v, w)$ if and only if (u, v, w) is permutation of (x, y, z).

Let $d : M^{(3)} \to [0, +\infty)$.

We consider the following conditions for such a map:

(M0) $d(x, x, x) = 0$, for any $x \in M$;

(M1) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$, for any $x, y, z, a \in M$;

(M2) $d(x, y, z) \leq d(x, a, b) + d(y, a, b) + d(z, a, b)$, for any $x, y, z, a, b \in M$;

(Ms) $d(x, x, y) = d(x, y, y)$, for any $x, y \in M$.

Let ρ be a subset of $M^{(3)}$.

We consider the following conditions for such a set:

(E0) $(x, x, x) \in \rho$, for any $x \in M$;

(E1) $(x, y, a), (x, a, z), (a, y, z) \in \rho \Rightarrow (x, y, z) \in \rho$, for any $x, y, z, a \in M$;

(E2) $(x, a, b), (a, y, b), (a, b, z) \in \rho \Rightarrow (x, y, z) \in \rho$, for any $x, y, z, a, b \in M$.

Definition 1 ([1]). A subset ρ of $M^{(3)}$: satisfying (E0) and (E1), is called a $(3, 1)$-equivalence; satisfying (E0) and (E2) is called a $(3, 2)$-equivalence; and satisfying (E0), (E1) and (E2) is called a 3-equivalence on M.

Let $\rho_d = \{(x, y, z) | (x, y, z) \in M^{(3)}, d(x, y, z) = 0\}$.

If d satisfies (M0) and (M1), then $\rho = \rho_d$ is a $(3, 1)$-equivalence, and if d satisfies (M0) and (M2), then $\rho = \rho_d$ is a $(3, 2)$-equivalence on M.

Definition 2 ([1]). If d satisfies (M0) and (Mj) we say that d is a $(3, j, \rho)$-metric on M; and if d satisfies (M0), (Mj) and (Ms) we say that d is a $(3, j, \rho)$-symmetric on M.

If $\rho = \Delta = \{(x, x, x) | x \in M\}$, then we write $(3, j)$ instead of $(3, j, \Delta)$.

Let $x \in M$ and $\varepsilon > 0$.

We define an ε-ball with center x and radius ε by:

$B(x, x, \varepsilon) = \{y | y \in M, d(x, x, y) < \varepsilon\}$.

Let $\tau(N, d)$ be the topology on M defined by:

$U \in \tau(N, d)$ iff for each $x \in U$, there is $\varepsilon > 0$ such that $B(x, x, \varepsilon) \subseteq U$.

Definition 3 ([1]). We say that a topological space (M, τ) is $(3, j, \rho)$-N-metrizable if there is a $(3, j, \rho)$-metric d such that $\tau = \tau(N, d)$.

Let d be a $(3, j, \rho)$-metric on M, $j \in \{1, 2\}$. We consider three types of convergence for a sequence $(x_n)_{n=1}^{\infty}$ in M that can be defined for the map d.

Definition 4. We say that the sequence $(x_n)_{n=1}^{\infty}$ in M:

i) 1-converges to $x \in M$;

ii) 2-converges to $x \in M$;

iii) 3-converges to $x \in M$

if:

i) $d(x, x, x_n) \to 0$ as $n \to \infty$;

ii) $d(x, x_n, x_n) \to 0$ as $n \to \infty$;

iii) $d(x, x_n, x_m) \to 0$ as $n, m \to \infty$.

Remark 1. We will use the notation $x_n \xrightarrow{\tau} x$ for the usual convergence in topological spaces.

Definition 5 ([12]). A map $D : M^2 \to [0, +\infty)$ such that $d(x, y) = 0$ iff $x = y$ and $d(x, y) = d(y, x)$, for all $x, y \in M$, is called symmetric on M. A topological space (M, τ) is called symmetrizable if there is a symmetric D on M such that U is open iff for each $x \in U$, there is $\varepsilon > 0$ such that $T(x, \varepsilon) = \{y | D(x, y) < \varepsilon\} \subseteq U$.

2. MAIN RESULTS

Lemma 1. If $D : M^2 \to [0, +\infty)$ is a symmetric on M, then the maps $d_D : M^3 \to [0, +\infty)$ and $d_{\max} : M^3 \to [0, +\infty)$ defined by:

$$d_D(x, y, z) = \frac{D(x, y) + D(x, z) + d(y, z)}{2},$$

$$d_{\max}(x, y, z) = \max\{D(x, y), D(x, z), D(y, z)\}$$

are $(3, 1)$-symmetrics on M.

Proof. First we will prove that d_D is a $(3, 1)$-symmetric on M.

1° $d_D(x, y, z) = 0 \Leftrightarrow D(x, y) = D(y, z) = D(y, z) \Leftrightarrow x = y = z$.

2° Since D is a symmetric, $d_D(x, y, z) = d_D(x, z, y) = d_D(y, z, x)$ and $d_D(x, x, y) = d_D(x, y, y) = D(x, y)$.

3° $d_D(a, y, z) + d_D(x, a, z) + d_D(x, y, a) = \frac{D(a,y)+D(a,z)+D(y,z)}{2}$
$+\frac{D(x,a)+D(x,z)+D(a,z)}{2} + \frac{D(x,y)+D(x,a)+D(y,a)}{2}$
$\geq \frac{D(x,y)+D(x,x)+D(y,z)}{2} = d_D(x, y, z)$.

Next we will prove that $d_{\max}$ is a $(3, 1)$-symmetric on M.

1° If $d_{\max}(x, y, z) = 0$, then $D(x, y) = D(x, z) = 0$, i.e. $x = y = z$.

2° It is obvious that $d_{\max}(x, y, z) = d_{\max}(x, z, y) = d_{\max}(y, z, x)$ and $d_{\max}(x, x, y) = d_{\max}(x, y, y)$.

3° Since for all $a, x, y \in M$, $d_{\max}(x, y, a) \geq D(x, y)$, it follows that $d_{\max}(a, y, z) + d_{\max}(x, a, z) + d_{\max}(x, y, a) \geq \max\{D(x, y), D(x, z), D(y, z)\} = d_{\max}(x, y, z)$. $\square$

Theorem 1. For any $(3, 1, \rho)$-metric d on M:

a) 3-convergence implies 2-convergence;

b) if d is a $(3, 1, \rho)$-symmetric, then 1-convergence is equivalent to 2-convergence.

Proof. Follows directly from the definitions. $\square$

Theorem 2. If (M, D) is a metric space and (M, d_D) is the corresponding $(3, 1)$-metric space, then the following conditions are equivalent:

(1) the sequence $(x_n)_{n=1}^{\infty}$1-converges to $x \in M$;

(2) the sequence $(x_n)_{n=1}^{\infty}$2-converges to $x \in M$;

(3) the sequence $(x_n)_{n=1}^{\infty}$3-converges to $x \in M$.

Proof. From $d_D(x, x, x_n) = D(x, x_n) = d_D(x, x_n, x_n)$ and
$d_D(x, x_n, x_m) = \frac{D(x,x_n)+D(x,x_m)+D(x_n,x_m)}{2}$ it follows that the conditions (1), (2) and (3) are equivalent. Moreover, they correspond to the convergence with respect to the metric space (M, D). In general, no other implication is true for $(3, 1, \rho)$-metric. $\square$

Example 1. Let $M = \{0\} \cup \left\{\frac{1}{n} | n \in \mathbb{N}\right\}$ and let $D : M^2 \to [0, +\infty)$ be a symmetric defined by:

a) $D(x, x) = 0$, for any $x \in M$;

b) $D(\frac{1}{n}, 0) = D(0, \frac{1}{n}) = \frac{1}{n}$, for $n \in \mathbb{N}$;

c) $D(x, y) = 1$, for $x, y \in M \backslash \{0\}$ such that $x \neq y$.

Then the corresponding map $d_{\max}$ is a $(3,1)$-symmetric on M. The sequence $(\frac{1}{n})_{n=1}^{\infty}$1-converges to 0 and 2-converges to 0, but does not 3-converge to 0.

Example 2. Let $M = \{0\} \cup \left\{\frac{1}{n} | n \in \mathbb{N}\right\}$ and let $d : M^3 \to [0, +\infty)$ be defined by:

a) $d(x,x,x) = 0$, for any $x \in M$;

b) $d(x,y,z) = \frac{1}{n}$, if one of x,y,z is 0, and the other two are $\frac{1}{n}$;

c) $d(x,y,z) = 1$, otherwise.

Then the map d is a $(3,1)$-metric on M. The sequence $(\frac{1}{n})_{n=1}^{\infty}$2-converges to 0, but does not 1-converge to 0, and does not 3-converge to 0.

Example 3. Let $M = \{0\} \cup \left\{\frac{1}{n} | n \in \mathbb{N}\right\}$ and let $d : M^3 \to [0, +\infty)$ be defined by:

a) $d(x,x,x) = 0$, for any $x \in M$;

b) $d(x,y,z) = \frac{1}{n}$, if two of x,y,z are 0, and the third is $\frac{1}{n}$;

c) $d(x,y,z) = 1$, otherwise.

Then the map d is a $(3,1)$-metric on M. The sequence $(\frac{1}{n})_{n=1}^{\infty}$1-converges to 0, but does not 2-converge to 0 and does not 3-converge to 0.

Next, we show that in $(3,2)$-N-metrizable spaces these four types of convergence are equivalent.

Theorem 2. For any $(3,2,\rho)$-metric d on M and any sequence $(x_n)_{n=1}^{\infty}$in M the following conditions are equivalent:

(1) $(x_n)_{n=1}^{\infty}$1-converges to $x \in M$;

(2) $(x_n)_{n=1}^{\infty}$2-converges to $x \in M$;

(3) $(x_n)_{n=1}^{\infty}$3-converges to $x \in M$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows directly from the inequalities

$$\frac{d(y,y,x)}{2} \leq d(x,x,y) \leq 2d(y,y,x).$$

(3) $\Rightarrow$ (2). Let $d(x, x_n, x_m) \to 0$ as $n, m \to \infty$. Then for each $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $d(x, x_n, x_m) < \varepsilon$ for $n, m > n_0$. Specially, for $m = n \geq n_0$, $d(x, x_n, x_n) < \varepsilon$.

(2) $\Rightarrow$ (3). Let $d(x, x_n, x_n) \to 0$ as $n \to \infty$. Then $d(x, x_n, x_m) \leq d(x, x, x_n) + d(x, x, x_m) \leq 2d(x, x_n, x_n) + 2d(x, x_m, x_m) \to 0$ as $n, m \to \infty$. $\square$

Theorem 3. If $(M,\tau)$is a T_2-$(3,2,\rho)$ -N-metrizable space via $(3,2,\rho)$-metric d, then the sequence $(x_n)_{n=1}^{\infty}$in M, 1-converges to x iff $x_n \xrightarrow{\tau} x$.

Proof. Let $d(x,x,x_n) \to 0$ as $n \to \infty$, and let $U \in \tau$ such that $x \in U$. Then there is $\varepsilon > 0$ such that $B(x,x,\varepsilon) \subseteq U$. Since $d(x,x,x_n) \to 0$ as $n \to \infty$, there is $n_0 \in \mathbb{N}$ such that $d(x,x,x_n) < \varepsilon$ for $n \geq n_0$. So, $x_n \in B(x,x,\varepsilon) \subseteq U$ for $n \geq n_0$, i.e. $x_n \xrightarrow{\tau} x$.

Let $x_n \xrightarrow{\tau} x$ and let us suppose that $d(x,x,x_n) \not\to 0$ as $n \to \infty$. Then there are $\varepsilon > 0$ and subsequence $(x_{n_k})_{k=1}^{\infty}$of the sequence $(x_n)_{n=1}^{\infty}$, such that $d(x,x,x_{n_k}) \geq \varepsilon > 0$ for each $k \in \mathbb{N}$. Since (M,τ) is T_2-space and

$G = \{x\} \cup \{x_{n_i} | i \in \mathbb{N}\} \subseteq M$ is compact, it follows that G is closed. We will show that $(M \backslash G) \cup \{x\} = M \backslash (G \backslash \{x\})$ is open.

Let $y \in (M \backslash G) \cup \{x\}$. Then $y \in M \backslash G$ or $y = x$.

1° If $y \in M \backslash G$, then there is $\delta > 0$ such that $B(y, y, \delta) \subseteq M \backslash G$ (G is closed). So, $B(y, y, \delta) \subseteq (M \backslash G) \cup \{x\}$.

2° Let $y = x$. Then by the assumption for $\varepsilon > 0$ and the definition of G, it follows that for each $z \in G \backslash \{x\}$, $d(y, y, z) = d(x, x, z) \geq \varepsilon > 0$. This means that $B(y, y, \varepsilon) \subseteq (M \backslash G) \cup \{x\}$.

From 1° and 2° follows that $(M \backslash G) \cup \{x\} = M \backslash (G \backslash \{x\})$ is open, which is impossible since $G \backslash \{x\}$ is not closed set (x is in the closure of $G \backslash \{x\}$, but is not in $G \backslash \{x\}$). □

Theorem 1 and Theorem 2 together with the fact that a $(3, 2)$-N-metrizable space is a T_2-space ([2]), imply the following corollary.

Corollary 1. For any $(3, 2)$-N-metrizable space (M, τ) via $(3, 2)$-metric d, and any sequence $(x_n)_{n=1}^{\infty}$in M, the following conditions are equivalent:

(1) $(x_n)_{n=1}^{\infty}$1-converges to $x \in M$;

(2) $(x_n)_{n=1}^{\infty}$2-converges to $x \in M$;

(3) $(x_n)_{n=1}^{\infty}$3-converges to $x \in M$;

(4) $x_n \xrightarrow{\tau} x$. □

Next, we give sufficient conditions for symmetrizability.

For a $(3, j)$-metric d on M, $j \in \{1, 2\}$ and a sequence $(x_n)_{n=1}^{\infty}$in M we state two conditions.

(s_1) If $(x_n)_{n=1}^{\infty}$1-converges to $x \in M$, then $(x_n)_{n=1}^{\infty}$2-converges to $x \in M$.

(s_2) If $(x_n)_{n=1}^{\infty}$2-converges to $x \in M$, then $(x_n)_{n=1}^{\infty}$1-converges to $x \in M$.

Theorem 4. If the $(3, j)$-metric d, $j \in \{1, 2\}$, satisfies the condition (s_1), then the map $D_1 : M^2 \to [0, +\infty)$ defined by:

$D_1(x, y) = \max\{d(x, x, y), d(x, y, y)\}$

is a symmetric and $\tau_{D_1} = \tau(N, d)$.

Proof. We will consider only the case when $j = 1$. The case when $j = 2$ can be proved similarly. It is obvious that D_1 is symmetric on M. We will prove that for the topology τ_{D_1}induced by D_1, $\tau_{D_1} = \tau(N, d)$.

Let $U \in \tau(N, d)$ and $x \in U$. There is $\varepsilon > 0$ such that $B(x, x, \varepsilon) \subseteq U$. The inclusion $T_1(x, \varepsilon) = \{y | D_1(x, y) < \varepsilon\} \subseteq B(x, x, \varepsilon)$ follows directly from the definition of D_1. Hence, $U \notin \tau_{D_1}$.

Let $U \in \tau_{D_1}$ and $x \in U$. Then there is $\varepsilon > 0$ such that $T_1(x, \varepsilon) \subseteq U$. Let us suppose that $B(x, x, \delta) \cap (M \backslash U) \neq \emptyset$ for each $\delta > 0$. Then for each $n \in \mathbb{N}$ there is $x_n \in M \backslash U$ such that $d(x, x, x_n) < \frac{1}{n}$. Thus, $d(x, x, x_n) \to 0$ as $n \to \infty$, i.e. $D_1(x, x_n) \to 0$ as $n \to \infty$. So, there is $n_0 \in \mathbb{N}$ such that $x_n \in T_1(x, \varepsilon) \subseteq U$ for $n \geq n_0$, which is impossible. Thus there is $\delta > 0$ such that $B(x, x, \delta) \subseteq U$. Hence, $U \in \tau(N, d)$. □

Theorem 5. If the $(3, j)$-metric d, $j \in \{1, 2\}$, satisfies the condition (s_2), then the map $D_1 : M^2 \to [0, +\infty)$ defined by:

$D_2(x, y) = \min\{d(x, x, y), d(x, y, y)\}$

is a symmetric and $\tau_{D_2} = \tau(N, d)$.

Proof. Analogous to the proof of the Theorem 4. $\square$

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