Математички Билтен ISSN 0351-336X
32 (LVIII)
2008 (17-27)
Скопје, Македонија

ON INEQUALITIES COMPLEMENTARY TO JENSEN'S INEQUALITY

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Abstract. In this paper we give generalizations of two complementary inequalities proved by Pečarić and Mesihović. We also show that a generalization of Niculescu's inequality obtained by M. Dincă, S. Rădulescu and M. Bencze is a simple consequence of an older theorem proved by Pečarić and Mesihović. We give an example which shows that a conjecture stated by M. Dincă, S. Rădulescu and M. Bencze is incorrect.

1. Introduction

In paper [1] M. Dincă, S. Rădulescu and M. Bencze proved the following theorem.

THEOREM A. *Let $f:[a,b]\to\mathbb{R}$, $a<b$, be a differentiable and convex function and $a\le x_1\le x_2\le ...\le x_n\le b$. Then*

$$\sum_{k=1}^{n} f(x_k) - nf\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) \le \max_{1\le k\le n-1}\left\{kf(a)+(n-k)f(b)-nf\left(\frac{ka+(n-k)b}{n}\right)\right\}. \tag{1.1}$$

In the same paper the authors also stated this conjecture.

CONJECTURE A. *Let $f:[a,b]\to\mathbb{R}$, $a<b$, be a differentiable and convex function, $p_i>0$ $(i=1,...,n)$ and $0<a\le x_1\le x_2\le ...\le x_n\le b$. Then*

$$\frac{1}{P_n}\sum_{k=1}^{n} p_k f(x_k) - f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k\right) \le \max_{1\le k\le n-1}\left\{\frac{kf(a)+(n-k)f(b)}{n} - f\left(\frac{ka+(n-k)b}{n}\right)\right\},$$

where $P_n=\sum_{k=1}^{n} p_k$.

In paper [2] J. Pečarić and B. Mesihović proved the following two theorems.

Date: May 26, 2008.
2000 *Mathematics Subject Classification.* Primary 26D15.
Key words and phrases. Convex functions, Jensen's inequality, complementary inequalities.

THEOREM B. *Let* $f:[m,M]\to\mathbb{R}$, $m<M$, *be a function such that* $f''(x)>0$ *on* $[m,M]$, $p_i>0$ for $i\in I_n=\{1,2,...,n\}$, $x_i\in[m,M]$ *for* $i\in I_n$ *and* $P_n=\sum_{i\in I_n}p_i=1$. *Then*

$$\sum_{i=1}^{n}p_i f(x_i)-f\left(\sum_{i=1}^{n}p_i x_i\right)\le\max_{I\subset I_n}W(I)\le U \tag{1.2}$$

where

$$\begin{aligned}
P_I&=\sum_{i\in I}p_i,\ I\subset I_n,\\
W(I)&=P_I f(M)+(1-P_I)f(m)-f(P_I M+(1-P_I)m),\\
U&=\frac{f(M)-f(m)}{M-m}f'^{-1}\left(\frac{f(M)-f(m)}{M-m}\right)+\\
&\quad+\frac{Mf(M)-mf(m)}{M-m}-f\left(f'^{-1}\left(\frac{f(M)-f(m)}{M-m}\right)\right).
\end{aligned}$$

THEOREM C. *Let the conditions of Theorem B be fulfilled and*

$$m\le x_n\le...\le x_1\le M.$$

Then

$$\sum_{i=1}^{n}p_i f(x_i)-f\left(\sum_{i=1}^{n}p_i x_i\right)\le\max_{I_k\subset I_n}W(I_k)\le U,$$

where U *is defined as in Theorem B and*

$$\begin{aligned}
I_k&=\{1,2,...,k\},\quad P_k=\sum_{i=1}^{k}p_i\\
W(I_k)&=P_k f(m)+(1-P_k)f(M)-f(P_k m+(1-P_k)M).
\end{aligned}$$

In Section 2, as the main results of this paper, we give the versions of Theorem B and Theorem C in which the condition $f''(x)>0$ on $[m,M]$ is relaxed to the more natural condition that the function f is convex on $[m,M]$. Then, in Section 3, we show that for any function f which fulfills the condition $f''(x)>0$ on $[m,M]$, the result of Theorem A can be obtained as a special case of Theorem B. After that we give an example which shows that Conjecture A is incorrect.

2. Main results

Let $f:I\to\mathbb{R}$, $I\subset\mathbb{R}$, be a convex function. Then the inequalities

$$\begin{aligned}
\frac{f(w)-f(x)}{w-x}&\le\frac{f(w)-f(y)}{w-y}\le\frac{f(x)-f(y)}{x-y}\\
&\le\frac{f(x)-f(z)}{x-z}\le\frac{f(y)-f(z)}{y-z}
\end{aligned} \tag{2.1}$$

hold for any $w<x<y<z$ in $Int\,(I)$ (see for example [3]). We shall use this well known property of convex functions to prove the following lemma.

Lemma 1. *Let* $f:[m,M]\to\mathbb{R}$ *be a convex function on* $[m,M]$ *and* $z\in[m,M]$, $p\in(0,1)$ *fixed. Then the function* $\varphi:[m,M]\to\mathbb{R}$ *defined by*

$$\varphi(t)=(1-p)\,f(z)+pf(t)-f((1-p)\,z+pt)$$

for all $t\in[m,M]$, *has the following properties:*

i) for all $t\in[m,M]$

$$\varphi(t)\geq\varphi(z)=0; \tag{2.2}$$

ii) for all $t_1,t_2\in[m,z]$, *where* $t_1\leq t_2$

$$\varphi(m)\geq\varphi(t_1)\geq\varphi(t_2), \tag{2.3}$$

i.e. function φ *is nonincreasing on* $[m,z]$;

iii) for all $s_1,s_2\in[z,M]$, *where* $s_1\leq s_2$

$$\varphi(M)\geq\varphi(s_2)\geq\varphi(s_1), \tag{2.4}$$

i.e. function φ *is nondecreasing on* $[z,M]$.

Proof. *i)* We can easily see that $\varphi(z)=0$, and since f is a convex function we have $\varphi(t)\geq0$ for all $t\in[m,M]$.

ii) If $z=m$ then $[m,z]=\varnothing$ and inequalities (2.3) trivially hold, so let $z\in(m,M]$. If $t_1=m$ then the first inequality in (2.3) becomes equality, and if $t_1=z$ the same inequality follows from (2.2), so let $t_1\in(m,z)$. We have

$$\begin{aligned}
&\varphi(m)-\varphi(t_1)\\
&=(1-p)\,f(z)+pf(m)-f((1-p)\,z+pm)-\\
&-(1-p)\,f(z)-pf(t_1)+f((1-p)\,z+pt_1)\\
&=p\,(m-t_1)\,\frac{f(m)-f(t_1)}{m-t_1}+\\
&+p\,(t_1-m)\,\frac{f((1-p)\,z+pt_1)-f((1-p)\,z+pm)}{(1-p)\,z+pt_1-[(1-p)\,z+pm]}\\
&=p\,(t_1-m)\left[\frac{f((1-p)\,z+pt_1)-f((1-p)\,z+pm)}{(1-p)\,z+pt_1-[(1-p)\,z+pm]}-\frac{f(t_1)-f(m)}{t_1-m}\right].
\end{aligned}$$

Since $(1-p)\,z+pt_1\in(t_1,z)$, $(1-p)\,z+pm\in(m,z)$ and $(1-p)\,z+pt_1>(1-p)\,z+pm$, according to property (2.1) we obtain

$$\frac{f((1-p)\,z+pt_1)-f((1-p)\,z+pm)}{(1-p)\,z+pt_1-[(1-p)\,z+pm]}\geq\frac{f(t_1)-f(m)}{t_1-m},$$

so we can conclude

$$\varphi(m)-\varphi(t_1)\geq0.$$

Let $t_1,t_2\in[m,z]$, $t_1\leq t_2$. If $t_1=t_2$ we have $\varphi(t_1)=\varphi(t_2)$, so we consider the case $t_1<t_2$. If $t_2=z$, $\varphi(t_1)\geq\varphi(t_1)$ follows from *i*). Now, for $t_1,t_2\in[m,z)$

and $t_1 < t_2$ we have

$$\begin{aligned}
&\varphi(t_1) - \varphi(t_2) \\
&= (1-p) f(z) + pf(t_1) - f((1-p) z + pt_1) - \\
&- (1-p) f(z) - pf(t_2) + f((1-p) z + pt_2) \\
&= p(t_1 - t_2) \frac{f(t_1) - f(t_2)}{t_1 - t_2} + \\
&+ p(t_2 - t_1) \frac{f((1-p) z + pt_2) - f((1-p) z + pt_1)}{(1-p) z + pt_2 - [(1-p) z + pt_1]} \\
&= p(t_2 - t_1) \left[\frac{f((1-p) z + pt_2) - f((1-p) z + pt_1)}{(1-p) z + pt_2 - [(1-p) z + pt_1]} - \frac{f(t_2) - f(t_1)}{t_2 - t_1} \right].
\end{aligned}$$

Since $(1-p) z + pt_2 \in (t_2, z)$, $(1-p) z + pt_1 \in (t_1, z)$ and $(1-p) z + pt_2 > (1-p) z + pt_1$ according to (2.1) we obtain

$$\frac{f((1-p) z + pt_2) - f((1-p) z + pt_1)}{(1-p) z + pt_2 - [(1-p) z + pt_1]} \geq \frac{f(t_2) - f(t_1)}{t_2 - t_1},$$

so we can conclude

$$\varphi(t_1) - \varphi(t_2) \geq 0.$$

iii) If $z = M$ then $[z, M] = \varnothing$ and (2.4) trivially holds, so let $z \in [m, M)$. If $s_2 = M$ then the first inequality in (2.4) becomes equality, and if $s_2 = z$ the same follows from (2.2), so let $s_2 \in (z, M)$. We have

$$\begin{aligned}
&\varphi(M) - \varphi(s_2) \\
&= (1-p) f(z) + pf(M) - f((1-p) z + pM) - \\
&- (1-p) f(z) - pf(s_2) + f((1-p) z + ps_2) \\
&= p(M - s_2) \frac{f(M) - f(s_2)}{M - s_2} + \\
&+ p(s_2 - M) \frac{f((1-p) z + ps_2) - f((1-p) z + pM)}{(1-p) z + ps_2 - [(1-p) z + pM]} \\
&= p(M - s_2) \left[\frac{f(M) - f(s_2)}{M - s_2} - \frac{f((1-p) z + pM) - f((1-p) z + ps_2)}{(1-p) z + pM - [(1-p) z + ps_2]} \right].
\end{aligned}$$

Since $(1-p) z + pM \in (z, M)$, $(1-p) z + ps_2 \in (z, s_2)$ and$(1-p) z + pM > (1-p) z + ps_2$, according to (2.1) we know that

$$\frac{f((1-p) z + pM) - f((1-p) z + s_2)}{(1-p) z + pM - [(1-p) z + s_2]} \leq \frac{f(M) - f(s_2)}{M - s_2},$$

so we can conclude

$$\varphi(M) - \varphi(s_2) \geq 0.$$

Let $s_1, s_2 \in [z, M]$, $s_1 \leq s_2$. If $s_1 = s_2$ we have $\varphi(s_1) = \varphi(s_2)$, so we consider the case $s_1 < s_2$. If $s_1 = z$ $\varphi(s_1) \leq \varphi(s_2)$ follows from *i*). Now, for $s_1, s_2 \in (z, M]$

and $s_1 < s_2$ we have

$$\begin{aligned}
&\varphi(s_2) - \varphi(s_1) \\
&= (1-p) f(z) + pf(s_2) - f((1-p)z + ps_2) - \\
&- (1-p) f(z) - pf(s_1) + f((1-p)z + ps_1) \\
&= p(s_2 - s_1) \frac{f(s_2) - f(s_1)}{s_2 - s_1} + \\
&+ p(s_1 - s_2) \frac{f((1-p)z + ps_1) - f((1-p)z + ps_2)}{(1-p)z + ps_1 - [(1-p)z + ps_2]} \\
&= p(s_2 - s_1) \left[\frac{f(s_2) - f(s_1)}{s_2 - s_1} - \frac{f((1-p)z + ps_2) - f((1-p)z + ps_1)}{(1-p)z + ps_2 - [(1-p)z + ps_1]} \right].
\end{aligned}$$

Since $(1-p)z + ps_2 \in (z, s_2)$, $(1-p)z + ps_1 \in (z, s_1)$ and $(1-p)z + ps_2 > (1-p)z + ps_1$ according to (2.1) we know that

$$\frac{f((1-p)z + ps_2) - f((1-p)z + ps_1)}{(1-p)z + ps_2 - [(1-p)z + ps_1]} \le \frac{f(s_2) - f(s_1)}{s_2 - s_1},$$

hence we can conclude

$$\varphi(s_2) - \varphi(s_1) \ge 0,$$

and this completes the proof. □

Lemma 1 enables us to prove our first theorem which is a generalization of Theorem B.

Theorem 1. *Let $f : [m, M] \to \mathbb{R}$ be a convex function on $[m, M]$, $p_i > 0$ for $i \in I_n = \{1, 2, ..., n\}$, $x_i \in [m, M]$ for $i \in I_n$ and $P_n = \sum_{i \in I_n} p_i = 1$, where $n \ge 2$. Then*

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \le \max_{I \subset I_n} W(I)$$

where

$$P_I = \sum_{i \in I} p_i, \ I \subset I_n,$$

$$W(I) = P_I f(M) + (1 - P_I) f(m) - f(P_I M + (1 - P_I) m).$$

Proof. We define the function $\varphi_n : [m, M] \to \mathbb{R}$ by

$$\varphi_n(x) = \sum_{i=1}^{n-1} p_i f(x_i) + p_n f(x) - f\left(\sum_{i=1}^{n-1} p_i x_i + p_n x\right).$$

If we set

$$p = p_n \in (0, 1), \ t = x_n,$$

$$z = \frac{1}{1 - p_n} \sum_{i=1}^{n-1} p_i x_i \in [m, M],$$

then according to Lemma 1 we can deduce that φ_n reaches its maximum in $x = m$ or $x = M$. Let us denote with ξ_n such $x \in \{m, M\}$. Now we define a new function $\varphi_{n-1} : [m, M] \to \mathbb{R}$ by

$$\begin{aligned}\varphi_{n-1}(x) &= \sum_{i=1}^{n-2} p_i f(x_i) + p_{n-1}x + p_n f(\xi_n) \\ &\quad - f\left(\sum_{i=1}^{n-2} p_i x_i + p_{n-1}x + p_n \xi_n\right).\end{aligned}$$

Similarly as in the first step we conclude that φ_{n-1} reaches its maximum in $x = m$ or $x = M$. We can continue this process, and after n steps we obtain the n-tuple $\boldsymbol{\xi} = (\xi_1, ..., \xi_n)$, where $\xi_i \in \{m, M\}$ $(i = 1, ..., n)$, for which the inequality

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(\xi_i) - f\left(\sum_{i=1}^{n} p_i \xi_i\right)$$

holds. If we define

$$I = \{i \in I_n : \xi_i = M\},$$

we obtain

$$\begin{aligned}&\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \\ &\le P_I f(M) + (1 - P_I) f(m) - f(P_I M + (1 - P_I) m),\end{aligned}$$

which gives us

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \le \max_{I \subset I_n} W(I).$$

This completes the proof. □

The following theorem is in a similar way a generalization of Theorem C.

Theorem 2. *Let the conditions of Theorem 1 be fulfilled and*

$$m \le x_1 \le ... \le x_n \le M.$$

Then

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \le \max_{I_k \subset I_n} W(I_k),$$

where

$$\begin{aligned}I_k = \{1, 2, ..., k\}, \quad P_k &= \sum_{i=1}^{k} p_i \\ W(I_k) = P_k f(m) + (1 - P_k) f(M) &- f(P_k m + (1 - P_k) M).\end{aligned}$$

Proof. We define the function $\Phi : [m, M]^n \to \mathbb{R}$ by

$$\Phi(\boldsymbol{x}) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right).$$

Similarly as before, we consider the function $\varphi_2 : [m, M] \to \mathbb{R}$ defined by

$$\varphi_2(x) = p_1 f(x_1) + p_2 f(x) + \sum_{i=3}^{n} p_i f(x_i) - f\left(p_1 x_1 + p_2 x + \sum_{i=3}^{n} p_i x_i\right),$$

and we set

$$p = p_2 \in (0, 1), \; t = x_2, \quad z = \frac{1}{1 - p_2} \sum_{i=1, i \neq 2}^{n} p_i f(x_i).$$

Since z is a convex combination of $x_1, x_3, ..., x_n$, we have two possible cases:

$$z \in [x_1, x_2) \quad \text{or} \quad z \in [x_2, x_n].$$

If $z \in [x_2, x_n]$, we can apply Lemma 1, hence

$$\begin{aligned}\varphi_2(x_2) &\leq \varphi_2(x_1) \\ &= (p_1 + p_2) f(x_1) + \sum_{i=3}^{n} p_i f(x_i) \\ &\quad - f\left((p_1 + p_2) x_1 + \sum_{i=3}^{n} p_i x_i\right),\end{aligned}$$

and from this we can deduce

$$\Phi(\boldsymbol{x}) \leq (p_1 + p_2) f(x_1) + \sum_{i=3}^{n} p_i f(x_i) - f\left((p_1 + p_2) x_1 + \sum_{i=3}^{n} p_i x_i\right).$$

On the other hand, if $z \in [x_1, x_2)$, Lemma 1 can not be applied, so we must try a similar substitution on the right end of the interval $[x_1, x_n]$, i.e., we must try to replace x_{n-1} with x_n. To achieve this, we consider the function

$$\varphi_{n-1}(x) = \sum_{i=1}^{n-2} p_i f(x_i) + p_{n-1} f(x) + p_n f(x_n) - f\left(\sum_{i=1}^{n-2} p_i x_i + p_{n-1} x + p_n x_n\right)$$

and we set

$$p = p_{n-1} \in (0,1), \ t = x_{n-1},$$
$$z = \frac{1}{1-p_{n-1}} \sum_{i=1, i\neq n-1}^{n} p_i f(x_i).$$

Since z is a convex combination of $x_1, ..., x_{n-2}, x_n$, we have two possible cases:

$$z \in [x_1, x_{n-1}] \quad \text{or} \quad z \in (x_{n-1}, x_n].$$

If $z \in [x_1, x_{n-1}]$, we can apply Lemma 1, hence

$$\begin{aligned} \varphi_{n-1}(x_{n-1}) &\leq \varphi_{n-1}(x_n) \\ &= \sum_{i=1}^{n-2} p_i f(x_i) + (p_{n-1}+p_n) f(x_n) \\ &\quad - f\left(\sum_{i=1}^{n-2} p_i x_i + (p_{n-1}+p_n) x_n\right), \end{aligned}$$

and from this we can deduce

$$\begin{aligned} \Phi(\boldsymbol{x}) &\leq \sum_{i=1}^{n-2} p_i f(x_i) + (p_{n-1}+p_n) f(x_n) \\ &\quad - f\left(\sum_{i=1}^{n-2} p_i x_i + (p_{n-1}+p_n) x_n\right). \end{aligned}$$

On the other hand, if $z \in (x_{n-1}, x_n]$, Lemma 1 can not be applied, so we must analyze what happens if

$$z = \frac{1}{1-p_2} \sum_{i=1, i\neq 2}^{n} p_i f(x_i) \in [x_1, x_2) \tag{2.5}$$

and

$$z' = \frac{1}{1-p_{n-1}} \sum_{i=1, i\neq n-1}^{n} p_i f(x_i) \in (x_{n-1}, x_n]. \tag{2.6}$$

It can be easily shown that it is possible only for $n = 2$, and then we can carry out the following *"two point"* procedure. We start with $m \leq x_1 \leq x_2 \leq M$ and

$$\begin{aligned} \varphi(x) &= p_1 f(x) + p_2 f(x_2) - f(p_1 x + p_2 x_2), \\ p &= p_1, \quad t = x_1, \quad z = x_2. \end{aligned}$$

In this case, Lemma 1 gives us

$$\begin{aligned} \varphi(x_1) &\leq \varphi(m) \\ &= p_1 f(m) + p_2 f(x_2) - f(p_1 m + p_2 x_2). \end{aligned}$$

But if

$$\begin{aligned} &\varphi(x) = p_1 f(m) + p_2 f(x) - f(p_1 m + p_2 x), \\ &\quad p = p_2, \quad t = x_2, \quad z = m, \end{aligned}$$

we also have

$$\begin{aligned} \varphi(x_2) &\le \varphi(M) \\ &= p_1 f(m) + p_2 f(M) - f(p_1 m + p_2 M) \end{aligned}$$

i.e. we obtain

$$\begin{aligned} &p_1 f(m) + p_2 f(x_2) - f(p_1 m + p_2 x_2) \\ &\le p_1 f(m) + p_2 f(M) - f(p_1 m + p_2 M), \end{aligned}$$

which implies

$$\begin{aligned} &p_1 f(x_1) + p_2 f(x_2) - f(p_1 x_1 + p_2 x_2) \\ &\le p_1 f(m) + p_2 f(M) - f(p_1 m + p_2 M), \end{aligned}$$

and this means that statement of our theorem holds truth when $n = 2$. If $n \ge 3$, the situation descibed in (2.5) and (2.6) is not possible, so either we can replace x_2 with x_1 (if $z \in [x_2, x_n]$) or we can replace x_{n-1} with x_n (if $z \in [x_1, x_{n-1}]$). If we continue this process, after $n-2$ steps we end with only two points x_1 and x_n with weights P_k and $(1-P_k)$ respectively (for some $k \in \{1, 2, ..., n-1\}$), and we know that

$$\Phi(\boldsymbol{x}) \le P_k f(x_1) + (1-P_k) f(x_n) - f(P_k x_1 + (1-P_k) x_n).$$

Now we can apply our *"two point"* procedure and conclude

$$\Phi(\boldsymbol{x}) \le P_k f(m) + (1-P_k) f(M) - f(P_k m + (1-P_k) M).$$

From the above inequality follows

$$\Phi(\boldsymbol{x}) \le \max_{I_k \subset I_n} \{P_k f(m) + (1-P_k) f(M) - f(P_k m + (1-P_k) M)\},$$

and this completes the proof. □

3. Examples

Now we shall show that for any function f such that $f''(x) > 0$ on $[m, M]$, the result of Theorem A can be obtained easily from Theorem B.

If we choose:

$$\begin{aligned} &[m, M] = [a, b], \\ &\quad p_i = \frac{1}{n}, \ (i = 1, ...n), \end{aligned}$$

then from Theorem B we get

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\sum_{i=1}^{n}\frac{1}{n}x_i\right)$$
$$\leq \max_{1\leq k\leq n-1}\left\{\frac{k}{n}f(a) + \frac{(n-k)}{n}f(b) - f\left(\frac{k}{n}a + \frac{(n-k)}{n}b\right)\right\}.$$

Multiplying the above inequality by n we obtain (1.1). Let us note here that if $k=0$ or $k=n$ we have

$$\frac{k}{n}f(a) + \frac{(n-k)}{n}f(b) - f\left(\frac{k}{n}a + \frac{(n-k)}{n}b\right) = 0$$

so these cases can be omitted.

Remark 1. *As we can see, Theorem A can be obtained in the same way as a special case of Theorem 2 but under relaxed conditions.*

Now we give an example which shows that Conjecture A is not correct.

Example 1. *Let us consider the function* $f:[0,2]\to[0,+\infty)$ *defined by*

$$f(x) = x^2,\ x\in[0,2].$$

The function f *is convex and differentiable on* $[0,2]$. *It is obvious that the condition* $a>0$ *in Conjecture A is not necessary, so for the simplicity we can set* $[a,b] = [0,2]$. *For* $n=3$ *we will choose* $\boldsymbol{x} = (0,1,2)$ *and* $\boldsymbol{p} = \left(\frac{1}{2}, q, \frac{1}{2}-q\right)$, *where* $q\in\left(0,\frac{1}{2}\right)$. *Now we have*

$$\frac{1}{P_3}\sum_{k=1}^{3} p_k f(x_k) - f\left(\frac{1}{P_3}\sum_{k=1}^{3} p_k x_k\right)$$
$$= q + 4\left(\frac{1}{2}-q\right) - (1-q)^2 = -q^2 - q + 1.$$

On the other hand,

$$\max_{1\leq k\leq 2}\left\{\frac{kf(0) + (3-k)f(2)}{3} - f\left(\frac{k\cdot 0 + (3-k)\cdot 2}{3}\right)\right\}$$
$$= \max\left\{\frac{8}{3} - \frac{16}{9}, \frac{4}{3} - \frac{4}{9}\right\} = \max\left\{\frac{8}{9}\right\} = \frac{8}{9}.$$

This means that if Conjecture A was true, than for $[0,2]$ *the inequality*

$$-q^2 - q + 1 \leq \frac{8}{9}$$

would hold. This, of course, is not possible because the inequality

$$-q^2 - q + 1 > \frac{8}{9}$$

has real solutions $q\in\left(\frac{-9-3\sqrt{13}}{18}, \frac{-9+3\sqrt{13}}{18}\right)$ *of which some fulfill the condition* $q\in\left(0,\frac{1}{2}\right)$ *because* $\left(\frac{-9-3\sqrt{13}}{18}, \frac{-9+3\sqrt{13}}{18}\right)\cap\left(0,\frac{1}{2}\right) = \left(0, \frac{-9+3\sqrt{13}}{18}\right) \neq \varnothing$.

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