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TERNARY GROUPOID POWERS

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Dedicated to Professor Ǵorgi Čupona

Abstract. The notion of ternary groupoid powers is introduced and some of its properties are investigated. In particular, it is shown that the set E of ternary groupoid powers, under a suitably defined binary operation, is a cancellative monoid and that this monoid is free over the set of its irreducible elements.

In the paper [1] groupoid powers are considered to be elements of a term groupoid over the one element set $\{e\}$. Following this note, we introduce in the present work the notion of ternary groupoid powers and investigate some of their properties. The main results of the paper are Theorem 1 and Theorem 2 .

Let G be a nonempty set. The mapping $[\;] : G^3 \to G$ from the third cartesian power of G into G is called a *ternary operation*. The set G together with the ternary operation $[\;]$ is called a *ternary groupoid*, or shortly, 3-*groupoid* and will be denoted by $(G, [\;])$ or shortly by $\boldsymbol{G}$. The ternary groupoid of ternary terms over a given nonempty set X will be denoted by $\mathbf{T}_X = (T_X, [\;])$ and its elements by $t, u, v, \dots$ The groupoid $\mathbf{T}_X$ is *injective*, i.e. if $t_i, u_i \in T_X$, for $i = 1, 2, 3$, then $[t_1t_2t_3] = [u_1u_2u_3] \Rightarrow t_1 = u_1, t_2 = u_2, t_3 = u_3$.

For any term $t \in T_X$ we define the *length* $|\,t\,|$ of t and the *set of subterms* $P(t)$ of t in the following inductive way:

$$|\,x\,| = 1,\ |\,[t_1t_2t_3]\,| = |\,t_1\,| + |\,t_2\,| + |\,t_3\,|$$

$$P(x) = \{x\},\ P([t_1t_2t_3]) = \{[t_1t_2t_3]\} \cup P(t_1) \cup P(t_2) \cup P(t_3),$$

for any $x \in X$ and any $t_1, t_2, t_3 \in T_X$.

If X is an one-element set $\{e\}$, then we write $\boldsymbol{E} = (E, [\;])$ instead of $\mathbf{T}_{\{e\}} = (T_{\{e\}}, [\;])$. Note that $E = \{e, [\overset{3}{e}], [ee[\overset{3}{e}]], [e[\overset{3}{e}]e], [[\overset{3}{e}]ee], \dots\}$, where $[\overset{3}{e}]$ stands for $[eee]$. Its elements are called *ternary groupoid powers* and are denoted by $f, g, h, \dots$

The number of ternary groupoid powers of length $2n+1$ is denoted by $\delta(2n+1)$, where n is any nonnegative integer. By induction on length it can be obtained the

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following recurrent formula:

$$\delta(1) = 1, \quad \delta(2n+1) = \Sigma\, \delta(i)\delta(j)\delta(k),$$

where (i, j, k) is an ordered 3-tuple of odd positive integers such that $i + j + k = 2n + 1$, for $n \geq 1$.

By a result of P. Hall ([3]), $\delta(2n+1)$ can be obtained by the explicit formula

$$\delta(2n+1) = \frac{(3n)!}{(2n+1)!\, n!}.$$

Let $\boldsymbol{G} = (G, [\ ])$ be a 3-groupoid and $\mathbf{E} = (E, [\ ])$ be the ternary groupoid of ternary terms over the set $\{e\}$. Every $f \in E$ induces a transformation $f^{\boldsymbol{G}} : G \to G$, called an *interpretation* of f in $\boldsymbol{G}$, defined by:

$$(\forall a \in G)\ \ f^{\boldsymbol{G}}(a) = \varphi_a(f),$$

where $\varphi_a : E \to G$ is the homomorphism from $\mathbf{E}$ into $\boldsymbol{G}$, such that $\varphi_a(e) = a$. In other words, for any $f, g, h \in E$,

$$e^{\boldsymbol{G}}(a) = a,\ \ [fgh]^{\boldsymbol{G}}(a) = [f^{\boldsymbol{G}}(a) g^{\boldsymbol{G}}(a) h^{\boldsymbol{G}}(a)]. \tag{1}$$

We will usually write $f(a)$ instead of $f^{\boldsymbol{G}}(a)$ when $\boldsymbol{G}$ is understood.

By induction on the length of f, for any $f \in E$ and $t \in T_X$ it is shown that the following proposition holds.

Proposition 1. *If $t \in T_X$ and $f \in E$, then:*
a) $|\, f(t)\,| = |\, f\,| \cdot |\, t\,|$ *b)* $t \in P(f(t))$.

Proof. a) If $|\, f\,| = 1$, then $f = e$, and $|\, f(t)\,| = |\, e(t)\,| = |\, t\,| = 1 \cdot |\, t\,| = |\, f\,| \cdot |\, t\,|$. If $|\, f\,| = 3$, then $f = [\overset{3}{e}]$, and $|\, f(t)\,| = |\, [\overset{3}{e}](t)\,| = |\, [\, e(t)e(t)e(t)\,]\,| = |\, [\, ttt\,]\,| = |\, t\,| + |\, t\,| + |\, t\,| = 3 \cdot |\, t\,| = |\, f\,| \cdot |\, t\,|$. Therefore, the statement is true for $|\, f\,| = 1$ and $|\, f\,| = 3$. Suppose that $|\, f(t)\,| = |\, f\,| \cdot |\, t\,|$ for any ternary groupoid power with odd length less than or equal to $2k - 1$ and let $|\, f(t)\,| = 2k + 1$. Then, $f = [\, f_1 f_2 f_3\,]$, where f_1, f_2, f_3 have odd lengths less than or equal to $2k - 1$, $|\, f_1\,| + |\, f_2\,| + |\, f_3\,| = |\, f\,|$ and

$|\, f(t)\,| = |\, [\, f_1 f_2 f_3\,](t)\,| = |\, [\, f_1(t) f_2(t) f_3(t)\,]\,| = |\, f_1(t)\,| + |\, f_2(t)\,| + |\, f_3(t)\,| =$
$= |\, f_1\,||\, t\,| + |\, f_2\,||\, t\,| + |\, f_3\,||\, t\,| = (|\, f_1\,| + |\, f_2\,| + |\, f_3\,|) \cdot |\, t\,| = |\, f\,| \cdot |\, t\,|$.

b) If $|\, f\,| = 1$, then $f = e$, $f(t) = e(t) = t$, and $t \in P(t) = P(f(t))$. If $|\, f\,| = 3$, then $f(t) = [\, eee\,](t) = [\, e(t)e(t)e(t)\,] = [\, ttt\,]$, and $P(f(t)) = P([\, ttt\,]) = \{[\, ttt\,]\} \cup P(t)$. From the fact that $t \in P(t)$, it follows that $t \in P(f(t))$. Suppose that the claim is true for any $f \in E$, such that $|\, f\,| = 1, 3, \ldots, 2k - 1$. Then, $f = [\, f_1 f_2 f_3\,]$, where f_1, f_2, f_3 have odd lengths less than or equal to $2k - 1$ and $t \in P(f_i(t))$ for $i = 1, 2, 3$. By this, since $P(f(t)) = P([\, f_1(t) f_2(t) f_3(t)\,]) = \{f(t)\} \cup P(f_1(t)) \cup P(f_2(t)) \cup P(f_3(t))$, it follows that $t \in P(f(t))$. □

By induction on the length of f, injectivity of $\mathbf{T}_X$ and Prop.1 a), it can be shown that the following proposition holds.

Proposition 2. *Let f and g be ternary groupoid powers and $t, u \in T_X$. Then:*
a) $f(t) = f(u) \ \Rightarrow\ t = u$. *b)* $f(t) = g(t) \ \Rightarrow\ f = g$.
c) $f(t) = g(u) \ \wedge\ (|\, f\,| = |\, g\,| \ \vee\ |\, t\,| = |\, u\,|) \ \Leftrightarrow\ f = g \ \wedge\ t = u$. □

Proposition 3. *Let f and g be ternary groupoid powers and $t, u \in T_X$. Then:*
$$f(t) = g(u) \ \wedge \ |\,t\,| > |\,u\,| \ \Leftrightarrow \ (\exists\,!\, h \in E \setminus \{e\}) \ (\, t = h(u) \ \wedge \ g = f(h)\,).$$

Proof. Let $f(t) = g(u)$ and $|\,t\,| > |\,u\,|$. If $|\,f\,| = 1$, than $f = e$, and therefore $t = g(u)$. If $h = g$, then $t = h(u)$ and $g = e(g) = e(h) = f(h)$. Clearly, $g \neq e$, because if the opposite is true, we would have $f(t) = u$ and then by Prop.1 a), $|\,f\,||\,t\,| = |\,u\,|$, that contradicts $|\,t\,| > |\,u\,|$.

Suppose that the statement is true for $|\,f\,| = 1, 3, \ldots, 2k-1$. If $|\,f\,| = 2k+1$, then $f = [\,f_1 f_2 f_3\,]$, where $|\,f_1\,|, |\,f_2\,|, |\,f_3\,|$ are less than or equal to $2k-1$. If $f(t) = g(u)$, then $|\,g\,| = |\,[g_1 g_2 g_3]\,| \geq 2k+1$, and $[\,f_1(t) f_2(t) f_3(t)\,] = [g_1(u) g_2(u) g_3(u)\,]$. From the fact that $\mathbf{T}_X$ is injective, it follows that $f_1(t) = g_1(u)$. By the inductive supposition, there is $h \in E \setminus \{e\}$, such that $t = h(u)$. Then $g(u) = f(t) = f(h(u)) = (f(h))(u)$ and thus by the Prop.2 b), it follows that $g = f(h)$.

If $h' \neq e$ is a ternary groupoid power that has the property $t = h'(u)$ and $g = f(h')$, then by $h'(u) = h(u)$ and Prop.2 b), we obtain that $h' = h$. Hence, $h \neq e$ is a unique ternary groupoid power with that property.

The converse is also true. Namely, if there is a unique ternary groupoid power $h \neq e$, such that $t = h(u)$ and $g = f(h)$, then $|\,t\,| = |\,h\,||\,u\,| > |\,u\,|$ and $g(u) = (f(h))(u) = f(h(u)) = f(t)$. □

The corresponding translations of the above properties that can be done when $\mathbf{T}_X = (T_X, [\,])$ is replaced by $\mathbf{E} = (E, [\,])$, are obvious.

Define a binary operation $\circ$ on the set E of ternary groupoid powers by:
$$f, g \in E \ \Rightarrow \ f \circ g = f(g). \tag{2}$$

The operation $\circ$ is well defined by the formula (1), and therefore, $(E, \circ)$ is a groupoid. We obtain an algebra $(E, [\ ], \circ)$ with one ternary and one binary operation such that for any $g, f_1, f_2, f_3 \in E$
$$e \circ g = g, \ \ [\,f_1 f_2 f_3\,] \circ g = [\,(f_1 \circ g)(f_2 \circ g)(f_3 \circ g))\,]. \tag{3}$$

Note that $\circ$ is right distributive with respect to the operation $[\,]$. We will show the following

Theorem 1. *The groupoid $(E, \circ)$ is a cancellative monoid.*

Proof. Let $f, g, h \in E$. By the induction on $|\,f\,|$, one can show that $(f \circ g) \circ h = f \circ (g \circ h)$. If $|\,f\,| = 1$, then $f = e$, and therefore, $(f \circ g) \circ h = (e \circ g) \circ h = g \circ h = e \circ (g \circ h) = f \circ (g \circ h)$. Suppose that the proposition is true for any $f \in E$ with odd length, less than or equal to $2k-1$. Let $|\,f\,| = 2k+1$ and $f = [\,f_1 f_2 f_3\,]$. Then f_1, f_2, f_3 have odd lengths, less than or equal to $2k-1$, and thus $(f_i \circ g) \circ h = f_i \circ (g \circ h)$, for $i = 1, 2, 3$. Hence,
$$\begin{aligned}&(f \circ g) \circ h = ([\,f_1 f_2 f_3\,] \circ g) \circ h = [\,(f_1 \circ g)(f_2 \circ g)(f_3 \circ g)\,] \circ h = \\ &= [\,((f_1 \circ g) \circ h)((f_2 \circ g) \circ h)((f_3 \circ g) \circ h)\,] = [\,(f_1 \circ (g \circ h))(f_2 \circ (g \circ h))(f_3 \circ (g \circ h))\,] = \\ &= [\,f_1 f_2 f_3\,] \circ (g \circ h) = f \circ (g \circ h).\end{aligned}$$

Clearly, $e \in E$ is the identity element for the operation $\circ$. Namely, for any $g \in E$, by (1) and (2), we obtain that $e \circ g = e(g) = g$, i.e. e is a left identity. By induction on $|\,g\,|$ one can show that e is a right identity. If $|\,g\,| = 1$, then $g = e$, and therefore, $g \circ e = e \circ e = e = g$. Suppose that $g \circ e = g$ holds, for any $g \in E$ with odd length, less than or equal to $2k-1$. Let $|\,g\,| = 2k+1$ and $g = [\,g_1g_2g_3\,]$. Then $g_i \circ e = g_i$, where g_i has an odd length, less than or equal to $2k-1$, for $i = 1, 2, 3$. Therefore, $g \circ e = [\,g_1g_2g_3\,] \circ e = [\,(g_1 \circ e)(g_2 \circ e)(g_3 \circ e)\,] = [\,g_1g_2g_3\,] = g$, i.e. e is a right identity.

Hence, $(E, \circ)$ is a semigroup with identity element, i.e. $(E, \circ, e)$ a monoid.

The monoid $(E, \circ, e)$ is left cancellatve, i.e. $f \circ g = f \circ h \;\Rightarrow\; g = h$. Namely, if $f = e$, then $g = h$. Suppose that implication holds for any $f \in E$ with odd length, less than or equal to $2k-1$. If $|\,f\,| = 2k+1$ and $f = [\,f_1f_2f_3\,]$, then f_i has an odd length, less than or equal to $2k-1$ and $f_i \circ g = f_i \circ h \;\Rightarrow\; g = h$, for $i = 1, 2, 3$. Thus, $f \circ g = [\,f_1f_2f_3\,] \circ g = [\,(f_1 \circ g)(f_2 \circ g)(f_3 \circ g)\,]$ and $f \circ h = [\,f_1f_2f_3\,] \circ h = [\,(f_1 \circ h)(f_2 \circ h)(f_3 \circ h)\,]$. By $f \circ g = f \circ h$ and the injectivity of $\mathbf{E} = (E, [\;])$ it follows that $f_i \circ g = f_i \circ h$, for $i = 1, 2, 3$. By the inductive supposition, we obtain that $g = h$.

That $(E, \circ, e)$ is right cancellative can be shown analogously. □

A ternary groupoid power f is said to be *irreducible* in $(E, \circ, e)$ if and only if

$$f \neq e \;\;\wedge\;\; (f = g \circ h \;\Rightarrow\; g = e \;\vee\; h = e). \tag{4}$$

A ternary groupoid power f is said to be *reducible* in $(E, \circ, e)$ if and only if there are $g, h \in E \setminus \{e\}$ such that $f = g \circ h$.

Proposition 4. *If the length $|\,f\,|$ of the ternary groupoid power f is a prime number, then f is irreducible in $(E, \circ, e)$.*

Proof. Let $f = g \circ h$. By Prop.1 we obtain that $|\,f\,| = |\,g\,| \cdot |\,h\,|$. If $|\,f\,|$ is a prime number, then $|\,g\,| = 1$ or $|\,h\,| = 1$, i.e. (4) holds. □

Proposition 5. *If g, h, p_1, p_2 are ternary groupoid powers, such that p_1 and p_2 are irreducible in $(E, \circ, e)$ and if $g \circ p_1 = h \circ p_2$, then $g = h$ and $p_1 = p_2$.*

In other words: If $f \in E \setminus \{e\}$ is reducible, then it has a uniquely determined left and right divisors and the right divisor is irreducible.

Proof. If $|\,g\,| = 1$, then $g = e$ and $p_1 = g \circ p_1 = h \circ p_2$. From the irreducibility of p_1 and p_2 we obtain that $h = e$. Thus, $g = h$ and $p_1 = p_2$. Suppose that the proposition is true for $|\,g\,| \leq 2k+1$, i.e.

$$|\,g\,| \leq 2k+1 \;\;\wedge\;\; g \circ p_1 = h \circ p_2 \;\Rightarrow\; g = h \;\wedge\; p_1 = p_2.$$

Let $|\,g\,| = 2k+3$. (Clearly, $h \neq e$. Namely, if $h = e$, then we would have $g \circ p_1 = h \circ p_2 = e \circ p_2 = p_2$, and that contradicts the assumption that p_2 is irreducible.) Then $g = [\,g_1g_2g_3\,]$ and $h = [\,h_1h_2h_3\,]$, where g_1, g_2, g_3 have odd lengths less than or equal to $2k+1$. From $g \circ p_1 = [\,(g_1 \circ p_1)\,(g_2 \circ p_1)\,(g_3 \circ p_1)\,]$, $h \circ p_2 = [\,(h_1 \circ p_2)\,(h_2 \circ p_2)\,(h_3 \circ p_2)\,]$ and $g \circ p_1 = h \circ p_2$, and by the injectivity of $(E, [\;])$, it follows that $g_i \circ p_1 = h_i \circ p_2$, $i = 1, 2, 3$. By the inductive supposition we obtain that $g_i = h_i$, $i = 1, 2, 3$ and $p_1 = p_2$. Thus, $g = h$ and $p_1 = p_2$. □

The monoid $(E, \circ, e)$ of ternary groupoid powers is characterized by the following property.

Proposition 6. *For every $f \in E \setminus \{e\}$ in the monoid $(E, \circ, e)$ of ternary groupoid powers, there is a uniquely determined sequence of irreducible elements $p_1, \dots, p_n$, such that $f = p_1 \circ p_2 \circ \cdots \circ p_n$, or $f = p_1$, if f is irreducible.*

Proof. If f is irreducible or the length $|\, f \,|$ of f is a prime number, then the statement is clear. If f is reducible, then by the Prop.5, there is a uniquely determined pair (g_1, p) of ternary groupoid powers, such that $f = g_1 \circ p$ and p is irreducible. The same discussion can be repeated for g_1, too. This procedure will end after finite number of steps, because $|\, f \,| > |\, g_1 \,| > \dots$ □

By Prop.6 it follows that the set P of irreducible elements in $(E, \circ, e)$ is a generating set for $(E, \circ, e)$. Clearly, P is a countable set. The monoid $(E, \circ, e)$ has the universal mapping property for the class of monoids over P. Namely, let $(M, \cdot, 1_M)$ be a monoid and let $\lambda : P \to M$ be a mapping. Define a mapping $\varphi : E \to M$ by: $\varphi(e) = 1_M$, $\varphi(p_i) = \lambda(p_i)$ for any $p_i \in P$, and for any reducible $f \in E$, $\varphi(f) = \varphi(p_1 \circ p_2 \circ \cdots \circ p_n) = \lambda(p_1) \cdot \lambda(p_2) \cdot \cdots \cdot \lambda(p_n)$. Clearly, φ is an extension of λ and it is easily shown that φ is a homomorphism from $(E, \circ, e)$ into $(M, \cdot, 1_M)$. Therefore, $(E, \circ, e)$ has the the universal mapping property for the class of monoids over P. Thus, we have shown the following

Theorem 2. *The monoid $(E, \circ, e)$ of ternary groupoid powers is free over the countable set of irreducible elements in $(E, \circ, e)$.* □

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