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EXTENDING KANNAN AND CHATTERJEA THEOREMS IN 2-BANACH SPACES BY USING SEQUENTIALY CONVERGENT MAPPINGS

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Abstract. S. Gähler ([8]), defined a 2-normed and A. White ([1]) a 2-banach space Further, the contractive mapping in 2-normed space in [4] is defined by in P. K. Hatikrishnan, K. T. Ravindran. In [2] and [10], there are given generalizations of Kannan ([5]) and Chatterjea ([9]) fixed points theorems in 2-Banach spaces. In this paper by using sequentially convergent mappings, the results given in [2] and [10] will be generalized in 2-Banach spaces. They might also be considered as generalizations of the results given in [11].

1. INTRODUCTION

S. Gähler ([8]), 1965 defined a 2-normed space, and White ([1]), 1968, a 2-Banach space. 2-Banach spaces are focus of interest of many authors, and certain results can be seen in [6]. Furthermore, analogously as in the normed spaces, P. K. Hatikrishnan and K. T. Ravindran in [4] defined a contractive mapping in 2-normed space as following.

Definition 1 ([4]). Let $(L,\|\cdot,\cdot\|)$ be a real 2-normed space. The mapping $S:L\to L$ is said to be contraction if $\lambda\in[0,1)$ exists so that $\|Sx-Sy,z\|\leq\lambda\|x-y,z\|$, for all $x,y,z\in L$, holds true.

Hatikrishnan and Ravindran in [4], proved that a contractive mapping has a unique fixed point in a closed and bounded subset of 2-Banach space. Furthermore, M. Kir and H. Kiziltunc in [2] proved that if L is 2-Banach space and for $\alpha\in(0,\frac{1}{2})$, $S:L\to L$ satisfies one of the following conditions

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$$\| Sx - Sy, z \| \le \alpha(\| x - Sx, z \| + \| y - Sy, z \|) \text{, for all } x, y, z \in L \tag{1}$$

or

$$\| Sx - Sy, z \| \le \alpha(\| x - Sy, z \| + \| y - Sx, z \|) \text{, for all } x, y, z \in L \tag{2}$$

then, S has a unique fixed point in L. The case where the condition (1) is satisfied is actually a generalization of Kannan's Theorem and the case where the condition (2) is satisfied is a generalization of Chatterjea's Theorem. Furthermore, the generalizations of M. Kir and H. Kiziltunc results and their consequences are reviewed in [10]. Next, we will make a generalization of the above mentioned results, by using sequentially convergent mappings, defined as following.

Definition 2. Let $(L, \| \cdot, \cdot \|)$ be a 2-normed space. A mapping $T : L \to L$ is said to be sequentially convergent if for each sequence $\{y_n\}$ the following condition is satisfied:

If $\{Ty_n\}$ converges, then $\{y_n\}$ is also converges.

2. EXTENSIONS OF KANNAN AND CHATTERJEA THEOREMS

Theorem 1. Let $(L, \| \cdot, \cdot \|)$ be a 2-Banach space, $S : L \to L$ and the mapping $T : L \to L$ is continuous, injection and sequentially convergent. If $\alpha > 0$, $\gamma \ge 0$, are such that $2\alpha + \gamma < 1$, for all $x, y, z \in L$, and

$$\| TSx - TSy, z \| \le \alpha(\| Tx - TSx, z \| + \| Ty - TSy, z \|) + \gamma \| Tx - Ty, z \|, \tag{3}$$

then, S has a unique fixed point and for each $x_0 \in L$ the sequence $\{S^n x_0\}$ converges to the above fixed point.

Proof. Let x_0 be any point in L and let the sequence $\{x_n\}$ be defined as $x_{n+1} = Sx_n$, $n = 0, 1, 2, \ldots$. The inequality (3) and the definition of $\{x_n\}$ imply that

$$\| Tx_n - Tx_{n+1}, z \| \le \alpha(\| Tx_{n-1} - Tx_n, z \| + \| Tx_n - Tx_{n+1}, z \|) + \gamma \| Tx_n - Tx_{n-1}, z \|$$

This holds true for each $n = 0, 1, 2, 3, \ldots$ and for each $z \in L$. So, the inequality above implies

$$\| Tx_n - Tx_{n+1}, z \| \le \lambda^n \| Tx_0 - Tx_1, z \| \tag{4}$$

for each $n = 0, 1, 2, 3, \ldots$ and each $z \in L$, $0 < \lambda = \frac{\alpha+\gamma}{1-\alpha} < 1$. Furthermore, (4) implies that for all $m, n \in \mathbf{N}$, $n > m$ and for each $z \in L$

$$\| Tx_n - Tx_m, z \| \le \tfrac{\lambda^m}{1-\lambda} \| Tx_0 - Tx_1, z \|,$$

holds true. The sequence $\{Tx_n\}$ is Caushy and L is a 2-Banach space. So, the sequence $\{Tx_n\}$ is convergent. Furthermore, the mapping $T: L \to L$ is sequentially convergent, so the sequence $\{x_n\}$ is convergent, i.e. it exists $u \in L$ so that $\lim\limits_{n\to\infty} x_n = u$.

Since T is continuous, $\lim\limits_{n\to\infty} Tx_n = Tu$ and

$$\begin{aligned} \| TSu - Tu, z \| &\le \| TSu - TS^n x_0, z \| + \| TS^n x_0 - TS^{n+1} x_0, z \| + \| TS^{n+1} x_0 - Tu, z \| \\ &\le \alpha(\| Tu - TSu, z \| + \| TS^{n-1} x_0 - TS^n x_0, z \|) + \gamma \| Tu - TS^{n-1} x_0, z \| \\ &\quad + \| TS^n x_0 - TS^{n+1} x_0, z \| + \| TS^{n+1} x_0 - Tu, z \| \\ &= \alpha(\| Tu - TSu, z \| + k \| Tx_{n-1} - Tx_n, z \|) + \gamma \| Tu - Tx_{n-1}, z \| \\ &\quad + \| Tx_n - Tx_{n+1}, z \| + \| Tx_{n+1} - Tu, z \|. \end{aligned}$$

holds true for each $n = 0,1,2,3,...$ and each $z \in L$.

For $n \to \infty$, the continuous of T and the properties of 2-norm imply that $\| TSu - Tu, z \| \le \alpha \| TSu - Tu, z \|$ holds true for each $z \in L$. Since, $\alpha < 1$, this implies that $\| TSu - Tu, z \| = 0$, for each $z \in L$, i.e. $TSu = Tu$. Finally, T is injection, so, $Su = u$, and S has a fixed point.

Let $u, v \in X$ be fixed points of S, i.e. $Su = u$ and $Sv = v$. Then, (3) implies that

$$\| Tu - Tv, z \| = \| TSu - TSv, z \| \le \alpha(\| Tu - TSu, z \| + \| Tv - TSv, z \|) + \gamma \| Tu - Tv, z \|$$

holds true for each $z \in L$. Since $\gamma < 1$, we get that $\| Tu - Tv, z \| = 0$, for each $z \in L$, i.e. $Tu = Tv$. Since T is an injection, $u = v$, so T has a unique fixed point. Finally, the arbitrarily of $x_0 \in L$ implies that for each $x_0 \in L$ the sequence $\{S^n x_0\}$ converges to the unique fixed point of S. ■

Consequence 1. Let $(L, \| \cdot, \cdot \|)$ be a 2-Banach space, $S: L \to L$ and the mapping $T: L \to L$ be continuous, injection and sequentially convergent. If $\lambda \in (0,1)$ and

$$\| TSx - TSy, z \| \le \lambda \sqrt[3]{\| Tx - TSx, z \| \cdot \| Ty - TSy, z \| \cdot \| Tx - Ty, z \|}$$

holds true for all $x, y, z \in L$, then S has a unique fixed point.

Proof. For $\alpha = \gamma = \frac{\lambda}{3}$, the arithmetic-geometric inequality mean and the theorem 1, directly imply the statement.. ■

Consequence 2. Let $(L,\|\cdot,\cdot\|)$ be a 2-Banach space, $S:L\to L$ and the mapping $T:L\to L$ be continuous, injection and sequentially convergent. If there exist $\alpha>0$, and $\gamma\geq 0$, so that $2\alpha+\gamma<1$ and

$$\|TSx-TSy,z\|\leq\alpha\frac{\|Tx-TSx,z\|^2+\|Ty-TSy,z\|^2}{\|Tx-TSx,z\|+\|Ty-TSy,z\|}+\gamma\|Tx-Ty,z\|,$$

holds true for all $x,y,z\in L$, then S has a unique fixed point.

Proof. The inequality sated in the condition implies (3), and the required statement is directly implied by the theorem 1. ■

Consequence 3. Let $(L,\|\cdot,\cdot\|)$ be a 2-Banach space, $S:L\to L$ and the mapping $T:L\to L$ be continuous, injection and sequentially convergent. If $\alpha\in(0,\frac{1}{2})$ and

$$\|TSx-TSy,z\|\leq\alpha(\|Tx-TSx,z\|+\|Ty-TSy,z\|)$$

holds true for all $x,y,z\in L$, then S has a unique fixed point.

Proof. For $\gamma=0$, and applying the theorem 1, we get the required statement. ■

Remark 1. The theorem 1 and the consequences 1 and 2, for $Tx=x$, imply the validity of the theorem 1 and the consequences 1 and 2, [10], and the consequence 3, imply the validity of the theorem 1, [2].

Theorem 2. Let $(L,\|\cdot,\cdot\|)$ be a 2-Banach space, $S:L\to L$ and the mapping $T:L\to L$ be continuous, injection and sequentially convergent. If $\alpha>0$, $\gamma\geq 0$, such that $2\alpha+\gamma<1$ and

$$\|TSx-TSy,z\|\leq\alpha(\|Tx-TSy,z\|+\|Ty-TSx,z\|)+\gamma\|Tx-Ty,z\| \tag{5}$$

holds true for all $x,y,z\in L$, then S has a unique fixed point and for each $x_0\in X$ the sequence $\{S^n x_0\}$ converges to the above fixed point.

Proof. Let x_0 be any point in L and the sequence $\{x_n\}$ be defined as the following $x_{n+1}=Sx_n$, $n=0,1,2,3,...$. The inequality (5) implies that

$$\begin{aligned}\|Tx_{n+1}-Tx_n,z\|&=\|TSx_n-TSx_{n-1},z\|\\ &\leq\alpha(\|Tx_n-TSx_{n-1},z\|+\|Tx_{n-1}-TSx_n,z\|)+\gamma\|Tx_n-Tx_{n-1},z\|\\ &=\alpha\|Tx_{n+1}-Tx_{n-1},z\|+\gamma\|Tx_n-Tx_{n-1},z\|\\ &\leq\alpha(\|Tx_{n+1}-Tx_n,z\|+\|Tx_n-Tx_{n-1},z\|)+\gamma\|Tx_n-Tx_{n-1},z\|\end{aligned}$$

holds true for each $n=0,1,2,3,...$ and each $z\in L$. The previous inequality implies that

$$\| Tx_n - Tx_{n+1}, z \| \le \lambda^n \| Tx_0 - Tx_1, z \| \tag{6}$$

holds true for each $n = 0,1,2,3,...$ and each $z \in L$, for $0 < \lambda = \frac{\alpha+\gamma}{1-\alpha} < 1$. Furthermore, by using (6), analogously as in the theorem 1is is proven that the sequence $\{x_n\}$ is convergent, i.e. it exists $u \in L$ so that $\lim_{n\to\infty} x_n = u$ and $\lim_{n\to\infty} Tx_n = Tu$. From the inequality (5), for each $n = 0,1,2,3,...$ and each $z \in L$,

$$\begin{aligned}
\| TSu - Tu, z \| &\le \| TSu - TS^n x_0, z \| + \| TS^n x_0 - TS^{n+1} x_0, z \| + \| TS^{n+1} x_0 - Tu, z \| \\
&\le \alpha(\| Tu - TS^n x_0, z \| + \| TS^{n-1} x_0 - TSu, z \|) + \gamma \| Tu - TS^{n-1} x_0, z \| \\
&\quad + \| TS^n x_0 - TS^{n+1} x_0, z \| + \| TS^{n+1} x_0 - Tu, z \| \\
&= \alpha(\| Tu - Tx_n, z \| + \| Tx_{n-1} - TSu, z \|) + \gamma \| Tu - Tx_{n-1}, z \| \\
&\quad + \| Tx_n - Tx_{n+1}, z \| + \| Tx_{n+1} - Tu, z \|.
\end{aligned}$$

holds true. Analogously, as in the theorem 1, we come to the conclusion that $Su = u$, i.e. u is a fixed point of S. Finally, if v is another fixed point of S, the inequality (5) implies that

$$\| Tu - Tu, z \| \le (2\alpha + \gamma) \| Tu - Tv, z \|,$$

holds true for each $z \in L$. This implies that $u = v$. ■

Consequence 4. Let $(L, \| \cdot,\cdot \|)$ be a 2-Banach space, $S : L \to L$ the mapping $T : L \to L$ be continuous, injection and sequentially convergent. If $\lambda \in (0,1)$ and

$$\| TSx - TSy, z \| \le \lambda \sqrt[3]{\| Tx - TSy, z \| \cdot \| Ty - TSx, z \| \cdot \| Tx - Ty, z \|}$$

for all $x, y, z \in L$, then S has a unique fixed point.

Proof. For $\alpha = \gamma = \frac{\lambda}{3}$, the arithmetic-geometric inequality mean and theorem 2 imply the validity of the statement above.■

Consequence 5. Let $(L, \| \cdot,\cdot \|)$ be a 2-Banach space, $S : L \to L$ and the mapping $T : L \to L$ be continuous, injection and sequentially convergent. If there exist $\alpha > 0,\ \gamma \ge 0$ such that $2\alpha + \gamma < 1$ and

$$\| TSx - TSy, z \| \le \alpha \frac{\| Tx - TSy, z \|^2 + \| Ty - TSx, z \|^2}{\| Tx - TSy, z \| + \| Ty - TSx, z \|} + \gamma \| Tx - Ty, z \|,$$

for all $x, y, z \in L$, then S has a unique fixed point.

Proof. The inequality given in the condition implies the inequality (5). Finally, the required statement is directly implied by the theorem 2. ■

Consequence 6. Let $(L,\|\cdot,\cdot\|)$ be a 2-Banach space, $S:L\to L$ and the mapping $T:L\to L$ be continuous, injection and sequentially convergent. If $\alpha\in(0,\frac{1}{2})$ and

$$\|TSx-TSy,z\|\leq\alpha(\|Tx-TSy,z\|+\|Ty-TSx,z\|)$$

holds true for all $x,y,z\in L$, then S has a unique fixed point.

Proof. For $\gamma=0$, and applying the theorem 2, we get the required statement. ■

Remark 2. The theorem 2 and the consequences 4 and 5, for $Tx=x$, imply the validity of the theorem 2 and the consequences 3 and 4, [10], also the consequence 6 confirm the validity of the theorem 2, [2].

3. EXTENSIONS OF KOPARDE-WAGHMODE THEOREM

Theorem 3. Let $(L,\|\cdot,\cdot\|)$ be a 2-Banach space, $S:L\to L$ and the mapping $T:L\to L$ be continuous, injection and sequentially convergent. If $\alpha>0$, $\gamma\geq 0$, such that $2\alpha+\gamma<1$ and

$$\|TSx-TSy,z\|^2\leq\alpha(\|Tx-TSx,z\|^2+\|Ty-TSy,z\|^2)+\gamma\|Tx-Ty,z\|^2 \tag{7}$$

holds true for all $x,y,z\in L$, then S has a unique fixed point and for each $x_0\in X$ the sequence $\{S^n x_0\}$ converges to the above fixed point.

Proof. Let x_0 be any point in L and let the sequence $\{x_n\}$ be defined as $x_{n+1}=Sx_n$, $n=0,1,2,3,...$. The inequality (7) implies the following

$$\begin{aligned}\|Tx_{n+1}-Tx_n,z\|^2&=\|TSx_n-TSx_{n-1},z\|^2\\&\leq\alpha(\|Tx_n-TSx_n,z\|^2+\|Tx_{n-1}-TSx_{n-1},z\|^2)+\gamma\|Tx_n-Tx_{n-1},z\|^2\\&=\alpha(\|Tx_n-Tx_{n+1},z\|^2+\|Tx_{n-1}-Tx_n,z\|^2)+\gamma\|Tx_n-Tx_{n-1},z\|^2,\end{aligned}$$

for each $n=0,1,2,3,...$. The previous inequality and the condition given in the theorem, imply that for $\lambda=\sqrt{\frac{\alpha+\gamma}{1-\alpha}}<1$ the following holds true

$$\|Tx_{n+1}-Tx_n,z\|^2\leq\lambda\|Tx_n-Tx_{n-1},z\|^2,$$

for each $n=0,1,...$ and for each $z\in L$. Furthermore, analogously as the theorem 1 and 2 we get that the sequence $\{x_n\}$ is convergent, i.e. there is $u\in L$ such that $\lim\limits_{n\to\infty}x_n=u$ and also the continuous of T imply that $\lim\limits_{n\to\infty}Tx_n=Tu$.

We will prove that u is a fixed point of S. Namely, by using the inequality (7), it is easy to prov that $\|TSu - Tu, z\| \le \sqrt{\alpha}\, \|TSu - Tu, z\|$, holds true for each $z \in L$. Since $\sqrt{\alpha} < 1$, we get that $\|TSu - Tu, z\| = 0$, holds true for each $z \in L$. Analogously as the proof of the theorem 1, we get that u is a fixed point of S.
If v is one other fixed point of S, then (7) implies $\|Tu - Tv, z\| \le \gamma \|Tu - Tv, z\|$, for each $z \in L$. Thus, analogously to the proof of theorem 1, $u = v$. ■

Consequence 7. Let $(L, \|\cdot,\cdot\|)$ be a 2-Banach space, $S : L \to L$ and the mapping $T : L \to L$ be continuous, injection and sequentially convergent. If $\alpha \in (0, \frac{1}{2})$ and

$$\|TSx - TSy, z\|^2 \le \alpha(\|Tx - TSy, z\|^2 + \|Ty - TSx, z\|^2),$$

for all $x, y, z \in L$, then S has a unique fixed point.
Proof. For $\gamma = 0$ in the theorem 3, we get the required statement. ■

Remark 3. For $Tx = x$, the theorem 3 and the consequence 7, we get the validity of the theorem 3 and the consequence 5, [10].

4. CONCLUSION

In our previus observations by using the sequential convergent mapping we generalized several results about fixed point in 2-Banach space ([10]). Naturally, we wonder if Kannan, Chatterjea and Koparde-Waghmode theorems about common fixed point of mappings defined in 2-Banach spaces might be generalized by using sequentially continuous mapping?

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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