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A CHARACTERIZATION OF STRICTLY CONVEX 2-NORMED SPACE

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Abstract. The terms of 2-norm and 2-normed are given by S. Gähler in the paper [10], Ch. Diminnie, S. Gahler and A. White ([3]) gave the term of strictly convex 2-normed spaces. This paper consists of several characterizations of strictly convex 2-normed spaces.

1. Introduction

Let L be a real vector space with dimension greater than 1 and $\|\cdot,\cdot\|$ be a real function on $L\times L$ such that:

a) $\|x,y\|=0$ if and only if the set $\{x,y\}$ is linearly dependent;

b) $\|x,y\|=\|y,x\|$, for all $x,y\in L$;

c) $\|\alpha x,y\|=|\alpha|\cdot\|x,y\|$, for all $x,y\in L$ and for each $\alpha\in\mathbf{R}$,

d) $\|x+y,z\|\leq\|x,z\|+\|y,z\|$, for all $x,y,z\in L$.

The function $\|\cdot,\cdot\|$ is said a 2-*norm* on L, and $(L,\|\cdot,\cdot\|)$ is said a *vector* 2-*normed* space ([10]). Some of the fundamental properties of the 2-norm are the following:

1. $\|x,y\|\geq 0$, for all $x,y\in L$ and
2. $\|x,y+\alpha x\|=\|x,y\|$, for all $x,y\in L$ and for each $\alpha\in\mathbf{R}$.

Let $n>1$ be a positive integer, L be a real vector space, $\dim L\geq n$ and $(\cdot,\cdot|\cdot)$ be a real function on $L\times L\times L$ such that

i) $(x,x|y)\geq 0$, for all $x,y\in L$ and $(x,x|y)=0$ if and only if x and y are linearly dependent;

ii) $(x,y|z)=(y,x|z)$, for all $x,y,z\in L$,

iii) $(x,x|y)=(y,y|x)$, for all $x,y\in L$;

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iv) $(\alpha x, y \mid z) = \alpha(x, y \mid z)$, for all $x, y, z \in L$ and for each $\alpha \in \mathbf{R}$; and

v) $(x + x_1, y \mid z) = (x, y \mid z) + (x_1, y \mid z)$, for all $x, x_1, y, z \in L$.

The function $(\cdot,\cdot \mid \cdot)$ is said a 2-*inner product*, and $(L,(\cdot,\cdot \mid \cdot))$ is said a 2-*pre-Hilbert space* ([1]).

The concepts of 2-norm and 2-inner product are two-dimensional analogous to the concepts of a norm and an inner product. R. Ehret proved ([7]), that if $(L,(\cdot,\cdot \mid \cdot))$ is a 2-pre-Hilbert space, then $\| x, y \| = (x, x \mid y)^{1/2}$ defines a 2-norm. Thus, we get a vector 2-normed space $(L, \|\cdot,\cdot\|)$ and moreover, for all $x, y, z \in L$ it is true that

$$(a, b \mid c) = \frac{\|a+b,c\|^2 - \|a-b,c\|^2}{4}, \tag{1}$$

$$\| x + y, z \|^2 + \| x - y, z \|^2 = 2(\| x, z \|^2 + \| y, z \|^2), \tag{2}$$

The equality (2) is actually two-dimensional analogy to the parallelogram equality and it is said parallelepiped equality. Further, if $(L, \|\cdot,\cdot\|)$ is a vector 2-normed space such that for all $x, y, z \in L$, (1) holds true, then (2) defines 2-inner product on L, whereby for all $a, b \in L$, $\| x, y \| = (x, x \mid y)^{1/2}$, holds true

Let z be a fixed non-null element of L, $V(z)$ be a subspace of L generated by z and L_z be a factor space $L / V(z)$. Let x_z be the class of equivalence of x with respect to $V(z)$. Clearly, L_z is a vector space in which the operations vector addition and scalar multiplication are defined as following $x_z + y_z = (x + y)_z$ and $\alpha x_z = (\alpha x)_z$. In [6] is proven that $\| x_z \|_z = \| x, z \|$ defines a norm on L_z.

Let $x, y \in L$ be non-null elements and let $V(x, y)$ be the subspace of L generated by the vectors x and y. The vector 2-normed space $(L, \|\cdot,\cdot\|)$ is called a strictly convex if $\| x, z \| = \| y, z \| = \| \frac{x+y}{2}, z \| = 1$ and $z \notin V(x, y)$, for $x, y, z \in L$, implies $x = y$ ([3]). Several characterizations of strictly convex 2-normed space are given in papers [1], [5] – [8], [12] – [14], [16], [17], [23] and [24], and a few of them will be stated in the following theorem.

Theorem 1. Let $(L, \|\cdot,\cdot\|)$ be a 2-normed space. The following claims are equivalent:

1) $(L, \|\cdot,\cdot\|)$ is strictly convex

2) For any non-null $z \in L$, the space L_z is strictly convex

3) If $\| x+y,z \| = \| x,z \| + \| y,z \|$ and $z \notin V(x,y)$, for $x,y,z \in L$ then $y = \alpha x$ for some $\alpha > 0$.
4) If $\| x-u,z \| = \alpha \| x-y,z \|$, $\| y-u,z \| = (1-\alpha) \| x-y,z \|$, $\alpha \in (0,1)$ and $z \notin V(x-u,y-u)$, then $u = (1-\alpha)x + \alpha y$.
5) if $\| x,z \| = \| y,z \| = 1$, $x \neq y$ and $z \notin V(x,y)$, for $x,y,z \in L$, then $\| \frac{x+y}{2}, z \| < 1$. ■

Example 1 ([3]). Let L be 2-preHilber space and $x,y,z \in L$ be such that $\| x,z \| = \| y,z \| = 1$, $x \neq y$ and $z \notin V(x,y)$. Then the parallelepiped equality implies

$$\| \frac{x+y}{2}, z \|^2 + \| \frac{x-y}{2}, z \|^2 = 1. \tag{3}$$

But, $x \neq y$ and $z \notin V(x,y)$, thus $\| \frac{x-y}{2}, z \| > 0$ and the equality (3) implies $\| \frac{x+y}{2}, z \| < 1$. Finally, thereby Theorem 1, L is strictly convex. ■

Example 2 ([18]). In the set of bounded series of real numbers l^∞

$$\| x,y \| = \sup_{\substack{i,j \in \mathbf{N} \\ i<j}} \left| \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \right|, \quad x = (x_i)_{i=1}^\infty, y = (y_i)_{i=1}^\infty \in l^\infty$$

defines a 2-norm. That is, $(l^\infty, \| \cdot,\cdot \|)$ is a real 2-normed space. The vectors

$$x = (1-\tfrac{1}{2}, 1-\tfrac{1}{2^2}, \dots, 1-\tfrac{1}{2^n}, \dots), \quad y = (0, 1-\tfrac{1}{2}, 1-\tfrac{1}{2^2}, \dots, 1-\tfrac{1}{2^{n-1}}, \dots) \text{ and}$$

$$z = (1,0,0,\dots,0,\dots)$$

satisfy $\| x,z \| = \| y,z \| = \| \frac{x+y}{2}, z \| = 1$ and $z \notin V(x,y)$, but $x \neq y$. Therefore, l^∞ is not strictly convex 2-normed space. ■

2. MAINS RESULTS

Let $x,y \in L$. The set

$$[x,y] = \{\alpha x + (1-\alpha)y \mid \alpha \in [0,1]\}$$

is said a *line segment* (*segment*) with end points x and y. The set

$$(x,y) = \{\alpha x + (1-\alpha)y \mid \alpha \in (0,1)\}$$

is said an *opened line segment* with end points x and y.

Corollary 1. Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space. The following claims are equivalent:

1) $(L,\|\cdot,\cdot\|)$ is strictly convex
2) If $\|x+y,z\|=\|x,z\|+\|y,z\|$ and $z\notin V(x,y)$, for $x,y,z\in L$, then the set $[x,y]=\{\alpha x+(1-\alpha)y \mid \alpha\in[0,1]\}$ is linearly dependent.

Proof. Let the condition 2) holds true. Since $x,y\in[x,y]$, the set $\{x,y\}$ is linearly dependent, i.e. it exists $\alpha\in\mathbf{R}$ such that $y=\alpha x$. If we substitute in the condition $\|x+y,z\|=\|x,z\|+\|y,z\|$ and consider that $z\notin V(x,y)$, we get $|1+\alpha|=1+|\alpha|$, which implies $\alpha>0$. Thus, $y=\alpha x$ for some $\alpha>0$, so Theorem 1 implies that L is strictly convex.
Let L be a strictly convex space. If $\|x+y,z\|=\|x,z\|+\|y,z\|$ and $z\notin V(x,y)$, then Theorem 1 implies that $y=\alpha x$ for some $\alpha>0$. Let $x_t=tx+(1-t)y$, for $t\in[0,1]$. Thus, $x_t=(t+(1-t)\alpha)x$, for $t\in[0,1]$ and thereby all $t,p\in[0,1]$ satisfy $t+(1-t)\alpha>0$ and $p+(1-p)\alpha>0$ we get

$$x_t=(t+(1-t)\alpha)x=\tfrac{t+(1-t)\alpha}{p+(1-p)\alpha}(p+(1-p)\alpha)x_p=\tfrac{t+(1-t)\alpha}{p+(1-p)\alpha}x_p,$$

So, the set $\{x_t,x_p\}$ is linearly dependent, for all $t,p\in[0,1]$. Thus, the set $[x,y]=\{\alpha x+(1-\alpha)y \mid \alpha\in[0,1]\}$ is linearly dependent. ■

Let L be a 2-normed space, $x,z\in L$ and $r>0$. The set

$$B_z(x,r)=\{y\in L \mid \|y-x,z\|<r\}$$

is called an *opened ball with respect to* z *centered at* x *and radius* r. If $x=0$ and $r=1$, then $B_z(0,1)$ is called *a unit opened ball with respect to* z. The set

$$B_z[x,r]=\{y\in L \mid \|y-x,z\|\le r\}$$

is called *a closed ball with respect to* z *centered at* x *and radius* r. If $x=0$ and $r=1$, then $B_z[0,1]$ is called *a unit closed ball with respect to* z. The set

$$S_z(x,r)=\{y\in L \mid \|y-x,z\|=r\}$$

is called *a sphere with respect to* z *centered at* x *and radius* r. If $x=0$ and $r=1$, then $S_z(0,1)$ is called *a unit closed sphere with respect to* z. Clearly,

$$B_z(x,r)\subseteq B_z[x,r] \text{ and } B_z[x,r]=B_z(x,r)\cup S_z(x,r).$$

Lemma 1. Let L be a 2-normed space and $x,y,z\in L$. If

$$\|x+y,z\|=\|x,z\|+\|y,z\|,$$

then, for all $t,s \geq 0$

$$\| tx+sy,z \| = t\| x,z \| + s\| y,z \|, \tag{4}$$

holds true. If $z \notin V(x,y)$, then $[\frac{x}{\|x,z\|},\frac{y}{\|y,z\|}] \subseteq S_z(0,1)$.

Proof. Let $0 \leq s \leq t$. Then the properties of the 2-norm imply

$$\begin{aligned} t\| x,z \| + s\| y,z \| &\geq \| tx+sy,z \| \\ &= \| t(x+y)-(t-s)y,z \| \\ &\geq \| t(x+y),z \| - \| (t-s)y,z \| \\ &= t\| x+y,z \| - (t-s)\| y,z \| \\ &= t\| x,z \| + t\| y,z \| - t\| y,z \| + s\| y,z \| \\ &= t\| x,z \| + s\| y,z \|, \end{aligned}$$

i.e. in this case the equality (4) holds true. Analogously can be considered a case for $0 \leq t \leq s$.

Let $z \notin V(x,y)$. If $\alpha \in [0,1]$, then the equality (4) implies

$$\| \alpha \frac{x}{\|x,z\|} + (1-\alpha)\frac{y}{\|y,z\|}, z \| = \frac{\alpha}{\|x,z\|}\| x,z \| + \frac{1-\alpha}{\|y,z\|}\| y,z \| = 1.$$

Therefore, $[\frac{x}{\|x,z\|},\frac{y}{\|y,z\|}] \subseteq S_z(0,1)$. ■

Theorem 2. The 2-normed space L is strictly convex if and only if $\| x,z \| = \| y,z \| = 1$ and $[x,y] \subseteq S_z(0,1)$ imply $x=y$.

Proof. Let L be a strictly convex 2-normed space and let the conditions $\| x,z \| = \| y,z \| = 1$ and $[x,y] \subseteq S_z(0,1)$ be satisfied. Then, $[x,y] \subseteq S_z(0,1)$, for $\alpha = \frac{1}{2}$ implies $\frac{x+y}{2} = \frac{1}{2}x + (1-\frac{1}{2})y \in S_z(0,1)$. That is $\| \frac{x+y}{2}, z \| = 1$. Moreover, since L is strictly convex, we get that $x=y$.

Conversely, let $\| x,z \| = \| y,z \| = 1$ and $[x,y] \subseteq S_z(0,1)$ imply $x=y$. Let's assume that $\| x+y,z \| = \| x,z \| + \| y,z \|$ and $z \notin V(x,y)$. Thus, Lemma 1 implies $[\frac{x}{\|x,z\|},\frac{y}{\|y,z\|}] \subseteq S_z(0,1)$, which according to assumption means $\frac{x}{\|x,z\|} = \frac{y}{\|y,z\|}$, i.e. $x = \frac{\|x,z\|}{\|y,z\|}y$. Finally, the Theorem 1 implies L is strictly convex 2-normed space.■

Theorem 3. A 2-normed space L is strictly convex if and only if the following condition is satisfied

$$x,y\in S_z(0,1),\ x\neq y \text{ implies} \quad\quad (5)$$
$$\|\alpha x+\beta y,z\|<1, \text{ for } \alpha,\beta>0 \text{ and } \alpha+\beta=1.$$

Proof. Let the condition (5) be satisfied and $\|x,z\|=\|y,z\|=1$, $x\neq y$ and $z\notin V(x,y)$. Then $x,y\in S_z(0,1)$ and since $\alpha=\beta=\frac{1}{2}$ we get that $\|\frac{x+y}{2},z\|<1$, which according to theorem 1 means that L is strictly convex.
Conversely, let's assume that there exist $x,y\in S_z(0,1), x\neq y$ and $\alpha,\beta>0$, $\alpha+\beta=1$ such that $\|\alpha x+\beta y,z\|=1$. The latter implies that there exist $x,y,z\in L$ such that $\|x,z\|=\|y,z\|=1$, $x\neq y$ and $\alpha,\beta>0$, $\alpha+\beta=1$ such that

$$\|\alpha x+\beta y,z\|=\|\alpha x,z\|+\|\beta y,z\|.$$

Since Lemma 1,

$$\|t(\alpha x)+s(\beta y),z\|=t\|\alpha x,z\|+s\|\beta y,z\|,$$

holds true for all $t,s\geq 0$. For $t=\frac{1}{2\alpha}$, $s=\frac{1}{2\beta}$, in the last equality, we get that there exist $x,y,z\in L$ such that

$$\|x,z\|=\|y,z\|=1,\ \ x\neq y \text{ and } \|\tfrac{x+y}{2},z\|=1.$$

The latter, according to theorem 1, means that the space L is not strictly convex. ■

In the purpose of the next characterization of strictly convex 2-normed space we will use the extremal points of the convex sets. Let C be a convex set into 2-normed space L. The point $z\in C$ is said to be an *extremal* (*end*) *point* for the set C if $z=tx+(1-t)y$, for some $t\in(0,1)$ and some $x,y\in C$ implies $x=y$.

Theorem 4. 2- normed space L is strictly convex if and only if for each $z\in L$ each point of the unit sphere with respect to z is an extremal point of the closed unit ball with respect to z.
Proof. Let L be strictly convex space. It will be proven that each point of the set $S_z(0,1)$ is an extremal point of the set $B_z[0,1]$. Let $u\in S_z(0,1)$ and let $u=tx+(1-t)y$ for some $t\in(0,1)$ and some $x,y\in B_z[0,1]$. Thereby $u\in S_z(0,1)$, it is true that $\|u,z\|=1$ and since $x,y\in B_z[0,1]$ we get that $\|x,z\|\leq 1, \|y,z\|\leq 1$. It will be proven that $\|x,z\|=\|y,z\|=1$ holds true. Indeed, in otherwise it holds that

$$1=\|u,z\|=\|tx+(1-t)y,z\|\leq t\|x,z\|+(1-t)\|y,z\|<1,$$

which is contradictory. Thus, $\| x+y,z \| \le \| x,z \| + \| y,z \| = 2$. At the end, it will be proven that $\| x+y,z \| = 2$ holds. Indeed, in otherwise it holds that

$$\begin{aligned} 1 &= \| u,z \| = \| tu+(1-t)u,z \| \\ &= \| t[tx+(1-t)y]+(1-t)[tx+(1-t)y],z \| \\ &= \| t^2x+t(1-t)(x+y)+(1-t)^2y,z \| \\ &< t^2+2t(1-t)+(1-t)^2=1, \end{aligned}$$

which is contradictory. So, $\| x,z \| = \| y,z \| = \| \frac{x+y}{2},z \| = 1$ and since L is strictly convex space, we get that $x=y$, i.e. u is an extremal point of the closed unit ball with respect to z.

Conversely, let's assume that $\| x,z \| = \| y,z \| = \| \frac{x+y}{2},z \| = 1$ and $z \notin V(x,y)$, for $x,y,z \in L$. Thus, the point $u=\frac{1}{2}x+\frac{1}{2}y$ is on the unit sphere with respect to z. Therefore, it is an extremal point of the closed unit ball with respect to z, that is $x=y$, i.e. L is strictly convex space. ■

Example 3. Over the vector space $\mathbf{C}_{[0,1]}$, of continuous functions on the interval $[0,1]$, the function $\| \cdot,\cdot \| : \mathbf{C}_{[0,1]} \times \mathbf{C}_{[0,1]} \to \mathbf{R}$ defined as

$$\| x,y \| = \max_{s,t \in [0,1]} \left| \begin{vmatrix} x(t) & x(s) \\ y(t) & y(s) \end{vmatrix} \right| .$$

is a 2-norm. So, $(\mathbf{C}_{[0,1]}, \| \cdot,\cdot \|)$ is 2-normed space. The functions

$$x(t)=1,\ y(t)=1-t,\ z(t)=t^2 \in \mathbf{C}_{[0,1]}$$

satisfy

$$\| x,z \| = \max_{s,t \in [0,1]} \left| \begin{vmatrix} 1 & 1 \\ t^2 & s^2 \end{vmatrix} \right| = \max_{s,t \in [0,1]} | s^2-t^2 | = 1 \text{ and}$$

$$\| y,z \| = \max_{s,t \in [0,1]} \left| \begin{vmatrix} 1-t & 1-s \\ t^2 & s^2 \end{vmatrix} \right| = \max_{s,t \in [0,1]} | s^2-t^2+st^2-ts^2 | = 1 .$$

Therefore, $x,y \in S_z(0,1)$. Further the functions $u(t)=\frac{1}{2}x(t)+\frac{1}{2}y(t)=1-\frac{t}{2}$ satisfy

$$\| u,z \| = \max_{s,t \in [0,1]} \left| \begin{vmatrix} 1-\frac{t}{2} & 1-\frac{s}{2} \\ t^2 & s^2 \end{vmatrix} \right| = \max_{s,t \in [0,1]} | s^2-t^2-\tfrac{1}{2}ts^2+\tfrac{1}{2}st^2 | = 1 .$$

Therefore $u \in S_Z(0,1)$ and since it is not an extremal point of $B_Z[0,1]$ we deduce that the space $(\mathbf{C}_{[0,1]}, \|\cdot,\cdot\|)$ is not strictly convex space. ■

In the purpose of the next characterization of strictly convex 2-normed space we will introduce the term of minimal point with respect to the set $M \subseteq L$.

The point $v \in L$ is said to be *minimal point with respect to the set* M if $\|u-m,z\| \leq \|v-m,z\|$, $z \notin V(u-M) = V(\{u-m \mid m \in M\})$, $u,z \in L$ and for each $m \in M$ implies that $u = v$.

Theorem 5. 2-normed space L is strictly convex if and only if for all $x,y \in L$ the points of the segment $[x,y] = \{tx+(1-t)y \mid t \in [0,1]\}$ are minimal with respect to the set $\{x,y\}$.

Proof. Let L be strictly convex space. Clearly, the points x and y are minimal with respect to the set $\{x,y\}$. Thus, let $v_t = tx+(1-t)y$, $t \in (0,1)$ be any point of the opened line segment $[x,y]$. Let's assume that for some $u \in L$ and $z \notin V(x,y)$

$$\|u-x,z\| \leq \|v_t - x,z\| = (1-t)\|x-y,z\| \tag{5}$$

$$\|u-y,z\| \leq \|v_t - y,z\| = t\|x-y,z\|. \tag{6}$$

hold true.

Further, the inequalities (5) and (6) imply

$$\begin{aligned}\|x-y,z\| &\leq \|x-u,z\| + \|u-x,z\| \\ &\leq (1-t)\|x-y,z\| + t\|x-y,z\| \\ &= \|x-y,z\|.\end{aligned}$$

Thus, the following equalities hold true

$$\|u-x,z\| = (1-t)\|x-y,z\|, \; \|u-y,z\| = t\|x-y,z\|$$

and since $t \in (0,1)$ and $z \notin V(x-u,y-u)$, theorem 1 implies that $u = tx+(1-t)y = v_t$. That is, v_t is minimal point with respect to the set $\{x,y\}$.

Conversely, let for all $a,b \in L$ the points of the segment $[a,b]$ be minimal for the set $\{a,b\}$. If $\|x,z\| = \|y,z\| = \|\frac{x+y}{2},z\| = 1$ and $z \notin V(x,y)$, then

$$\|0-x,z\| = \|x,z\| = \|\tfrac{x+y}{2},z\| \leq \|\tfrac{1}{2}x + \tfrac{1}{2}(-y) - x,z\|,$$

$$\|0-(-y),z\| = \|y,z\| = \|\tfrac{x+y}{2},z\| \leq \|\tfrac{1}{2}x + \tfrac{1}{2}(-y) - (-y),z\|$$

and $z \notin V(x,y) = V(0-x, 0-(-y))$.

But, the point $\frac{1}{2}x+\frac{1}{2}(-y)$ belongs to a segment $[x,-y]$. That is, it is minimal with respect to the set $\{x,-y\}$. So,

$$\frac{1}{2}x+\frac{1}{2}(-y)=0,$$

i.e. $x=y$. That is, L is strictly convex. ■

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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