Математички Билтен
Vol. 40(LXVI) No. 1
2016 (73-78)
Скопје, Македонија

ISSN 0351-336X (print)
ISSN 1857-9914 (online)
UDC: 517.925.4

ABOUT THE ZEROS AND THE OSCILLATORY CHARACTER OF THE SOLUTION OF ONE AREOLAR EQUATION OF SECOND ORDER WITH ANALYTIC COEFFICIENT

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Abstract In the paper, one linear areolar equation of second order with analytic coefficients is considered, regarding the zeroes and the oscillatory character of its general solution. In this equation the first derivative is missing. Some theorems will be proven and some examples will be given for different cases of the coefficient.

1. INTRODUCTION

The notion of the term complex number, complex variable and complex function $f(z)$ is a few centuries old and more than a century old is the idea for expanding the operations derivative and integral to a function of complex-conjugated variable**,** $\overline{z} = x - iy$.

In 1909, G.V. Kolosov [1], during his efforts to solve a problem from the theory of elasticity, has introduced the expressions

$$\frac{1}{2}[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) = \frac{\hat{d}W}{dz} \text{ and} \tag{1}$$

$$\frac{1}{2}[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) = \frac{\hat{d}W}{d\overline{z}} \tag{2}$$

known as operatory derivatives of a complex function $W = W(z) = u(x, y)$ $+ iv(x, y)$ from a complex variable $z = x + iy$ and $\overline{z} = x - iy$**,** respectively. The operator rules for these derivatives are given in the monograph of Г. Н. Положий [2] (pages 18-31). In the mentioned monograph, are also defined the so called operatory integrals

2010 *Mathematics Subject Classification.* 34M45, 35Q74.
Key words and phrases. areolar derivative, areolar equation, solution, zeros of solution, oscillatory character, analytic function.

$$\hat{\int} f(z)dz \quad \text{and} \quad \hat{\int} f(z)d\bar{z}$$

by $z = x + iy$ and $\bar{z} = x - iy$, respectively, from the complex function $f = f(z)$ in the area $D \subseteq \mathbb{C}$, where their operatory rules are proven as well, page 32 - 41.

2. Reasons for introducing this equation and formulation of the problem

In the theory of real functions a big role has the term oscillatory and especially the term periodical, as a direct consequence of the Newton's laws. The equation $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$ is one of the oldest differential equations and at the same time the equation of oscillatory processes (from stretching the spring pendulum, to rotating motion of bodies bounded by mutual action of gravitational forces).

In the case when k i.e. m is variable, and if we introduce here a general function $y(x)$ and a regulator of the appearance $a(x)$ we get a differential equation

$$y'' + a(x)y = 0 \tag{3}$$

which is called an equation of oscillations, if

1. $a(x) > 0$ and
2. $a(x)$ is big enough to cause oscillations, which is expressed analytically with the condition the integral $\int_0^\infty a(x)dx$ to be divergent.

Analogous to the equation (3) for the functions of two complex variables $W = W(z,\bar{z})$, would be the equation with areolar derivatives from second order

$$\frac{\hat{d}^2W}{d\bar{z}^2} + A(z,\bar{z})W = 0 \tag{4}$$

where $A(z,\bar{z})$ is a given function and $W(z,\bar{z}) = u(x,y) + iv(x,y)$ is an unknown function by the variables z and $\bar{z}$, which is a subject of analysis in this paper. Here, the derivative $\frac{\hat{d}^2W}{d\bar{z}^2} = \frac{\hat{d}}{d\bar{z}}\left(\frac{\hat{d}W}{d\bar{z}}\right)$, and the derivative $\frac{\hat{d}W}{d\bar{z}}$ is defined with (2). One of the questions raised here is the following: Is there an analogy with real oscillations and whether (4) can be called areolar equation of oscillations? Whether the solutions of the equation (4) have zeros and what is their nature?

3. MAIN RESULT

In [8], we tried to answer the simplest case of the equation (4), which is also the closest to the real oscillations, i.e. $A(z,\overline{z}) = K = \alpha + i\beta$ where K is a complex constant. There, we considered an areolar equation with constant coefficients

$$\frac{\hat{d}^2 W}{d\overline{z}^2} + (\alpha + i\beta)W = 0 \tag{5}$$

and some theorems were proven. Also, we have considered some examples for various cases of the coefficient K .

In this paper we are considering the equation (4), where $A(z)$ is an analytic function. Because of that fact, we have that $\frac{\hat{d}A(z)}{d\overline{z}} = 0.$ So, in the equation

$$\frac{\hat{d}^2 W}{d\overline{z}^2} + A(z)W = 0 \tag{6}$$

we can consider $A(z)$ as it is in the rank of "a generalized constant" in operatory vocabulary. So we can do the procedures that are maid in [8], also here to the equation (6). We are looking for a particular solution from the following form

$$W = e^{r\overline{z}} \tag{7}$$

where $r = r(A(z))$ and the solutions of the complex characteristic equation

$$r^2 + A(z) = 0 \tag{8}$$

are depending from $\left(x, y\right)$, i.e.

$$r_{1/2} = \sqrt{-(\alpha_{1/2}(x,y) + i\beta_{1/2}(x,y))} \tag{9}$$

Repeating the procedure as in [8], i.e. using some formulas from complex analysis and trigonometry, i.e. if we put $-(\alpha + i\beta) = \rho e^{i\theta}$, where $\rho = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \operatorname{arctg}\frac{\beta}{\alpha}$, for (9) we have

$$r_{1/2} = \sqrt{\sqrt{\alpha^2 + \beta^2}}\,[\cos\frac{\operatorname{arctg}\frac{\beta}{\alpha} + 2k\pi}{2} + i\sin\frac{\operatorname{arctg}\frac{\beta}{\alpha} + 2k\pi}{2}], k = 0,1$$

i.e.

$$r_1 = \sqrt[4]{\alpha^2 + \beta^2}\,[\cos\frac{\operatorname{arctg}\frac{\beta}{\alpha}}{2} + i\sin\frac{\operatorname{arctg}\frac{\beta}{\alpha}}{2}]$$

and

$$r_2 = \sqrt[4]{\alpha^2 + \beta^2}\,[\cos\frac{\operatorname{arctg}\frac{\beta}{\alpha} + 2\pi}{2} + i\sin\frac{\operatorname{arctg}\frac{\beta}{\alpha} + 2\pi}{2}].$$

If we use some of the trigonometric formulas that are useful here, with short transformations we get

$$r_1(x,y)=\sqrt{\frac{\sqrt{\alpha^2(x,y)+\beta^2(x,y)}+\alpha(x,y)}{2}}+i\sqrt{\frac{\sqrt{\alpha^2(x,y)+\beta^2(x,y)}-\alpha(x,y)}{2}} \tag{10}$$

and

$$r_2(x,y)=-\sqrt{\frac{\sqrt{\alpha^2(x,y)+\beta^2(x,y)}+\alpha(x,y)}{2}}-i\sqrt{\frac{\sqrt{\alpha^2(x,y)+\beta^2(x,y)}-\alpha(x,y)}{2}} \tag{11}$$

and according to this, the general solution of (6) will be:

$$W(z,\overline{z})=C_1(z)e^{r_1(z)\overline{z}}+C_2(z)e^{r_2(z)\overline{z}}. \tag{12}$$

We have proven the following

Theorem. The areolar equation (6) with analytic coefficient $A(z)$ has general solution (12), where the complex functions $r_1(z)$ and $r_2(z)$ are given with (10) and (11).

Lets see some examples.

Example 1. Lets consider the equation $\frac{\hat{d}^2W}{d\overline{z}^2}-z^2W=0$. The substitute $W=e^{r\overline{z}}$ gives us the equation $r^2-z^2=0$ where from we get that $r_{1/2}=\pm z=\pm(x+iy)$ so,

$$W_1=e^{r_1\overline{z}}=e^{(x+iy)(x-iy)}=e^{x^2+y^2}$$

$$W_2=e^{r_2\overline{z}}=e^{-(x+iy)(x-iy)}=e^{-(x^2+y^2)}=\frac{1}{e^{x^2+y^2}}$$

and now the general solution is

$$W(z,\overline{z})=C_1(z)e^{|z|^2}+C_2(z)e^{-|z|^2}=C_1(z)e^{z\cdot\overline{z}}+C_2(z)e^{-z\cdot\overline{z}}.$$

We can see that the zeroes can be only the common zeroes of the "constants" $C_1(z)$ and $C_2(z)$.

Example 2. For the equation $\frac{\hat{d}^2W}{d\overline{z}^2}+z^2W=0$ the characteristic equation is $r^2+z^2=0$ where from we get $r_{1/2}=\pm z\cdot\sqrt{-1}=\pm iz$

$$W(z,\overline{z})=C_1(z)e^{iz\cdot\overline{z}}+C_2(z)e^{-iz\cdot\overline{z}}=C_1(z)e^{i|z|^2}+C_2(z)e^{-i|z|^2}$$

or

$$W = \cos(x^2 + y^2)[C_1(z) + C_2(z)] + i\sin(x^2 + y^2)[C_1(z) - C_2(z)]$$

It is obvious that the zeroes of W can be only the common zeroes of $C_1(z)$ and $C_2(z)$.

Example 3. Lets consider a coefficient $A(z)$ without zeroes, for example $A(z) = e^{2z}$. Then, we have the equation $\frac{\hat{d}^2 W}{d\overline{z}^2} + e^{2z} W = 0$ and with the substitute $W = e^{r\overline{z}}$ we get $r^2 = -e^{2z}$ where from $r_{1/2} = \pm ie^z$

$$W_1 = e^{ie^z \overline{z}}, W_2 = e^{-ie^z \overline{z}}.$$

So, we have now:

$$W_1 = e^{e^x [y\cos y - x\sin y]}\{\cos[e^x (x\cos y + y\sin y)] + i\sin[e^x (x\cos y + y\sin y)]\}$$

$$W_2 = e^{-e^x [y\cos y - x\sin y]}\{\cos[e^x (x\cos y + y\sin y)] - i\sin[e^x (x\cos y + y\sin y)]\}$$

Zeroes for W_1 we will get for $\cos\lambda = 0$ and $\sin\lambda = 0$ for

$$\lambda = e^x (x\cos y + y\sin y),$$

and the zeroes for W_2 are the same. From $\cos\lambda = 0$, we get $\lambda = (2k-1)\frac{\pi}{2}$, and from $\sin\lambda = 0$, we get $\lambda = n\pi$. Since this functions are never equal, we have that $W(z, \overline{z})$ has no zeroes, except for $W = 0$, i.e. for $C_1(z) = C_2(z) = 0$.

4. CONCLUSION

In [8] we concluded that in the linear areolar equation from II order (5), the oscillatority exists in the solution both in the real and in the imaginary part and $W(z, \overline{z}) = 0$ has zeroes, where the signs of α and β in the coefficient $K = \alpha + i\beta$ does not has any influence on that oscillatority.

From the previous examples, we can conclude that for the considered areolar equation from II order (6), zeroes of its solutions can be only the common zeroes of the generalized constants $C_1(z)$ and $C_2(z)$ which is different from the Theorem 2 in [8].

We can conclude that in $\frac{\hat{d}^2 W}{d\overline{z}^2} + KW = 0$ if K is not a constant, but it is a "generalized constant" $A(z)$, then the formulated theorem 2 in [8] no longer stands.

References

[1] Г. В. Колосов, *Об одном приложении теории функции комплесного переменного к плоское задаче математическои упругости*, 1909

[2] Г. Н. Положии, *Обопштение теории аналитических фукции комплесного переменного*, Издателство Киевского Университета, 1965

[3] S. B. Bank, I. Laine, *On the oscillation theory of* $f'' + A(z)f = 0$, *where* $A(z)$ *is entire*. Trans.Amer.Math.Soc., 273, (1982). No1, 351-363

[4] Б. Илиевски, *Линеарни ареоларни равенки (Контурна интеграција. Специјални функции од две комплексни променливи. Ареоларни Лапласови трансформации)*, Докторска дисертација, Скопје,1992

[5] Д. Димитровски, Б. Илиевски, С. Брсакоска и други: *Равенка Векуа со аналитички коефициенти*, Специјални изданија на Институтот за математика при ПМФ на Универзитетот „Св. Кирил и Методиј“- Скопје, 1997 год.

[6] D. Dimitrovski, M. Rajović, R. Stoiljković, *The generalization of the I. N. Vecua equation with the analytic coefficients*, Filomat No. 11 (1997), 29-32 (MR1619486)

[7] M. Rajović, D. Dimitrovski, R. Stojiljković, *Elemental solution of Vecua equation with analytic coefficients*, Bul. Ştiinţ. Univ. Politeh. Timişoara Ser. Mat. Fiz. 41(55) , no. 1 (1996), 14-21 MR1606858

[8] S. Brsakoska, *About the zeros and the oscillatory character of the solution of one areolar equation of second order with constant coefficient*, Mat. Bilten. Vol. 40, No. 1 (2016), 55–62

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