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INVARIANT THEOREMS IN EUCLIDEAN GEOMETRY WITH RESPECT TO CONICS

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Abstract. Research is usually preceded by natural aspiration to discover new knowledge based on well-known facts. Many scientific facts are known for centuries, however new peculiarities are discovered by contemporary means. Information technologies together with abundant arsenal of knowledge and skills for effective application initiate specific way of thinking. The dynamic geometric software turns out to be a basic instrument to study objects from the Euclidean geometry. The present paper uses the possibilities of "THE GEOMETER'S SKETCHPAD" (GSP) in the generalization of some classic and some not very popular theorems after analyzing basic properties of the objects under study.

1. INTRODUCTION

In the plane of a given triangle ABC some special points are determined, which characterize various properties of the triangle. Such points are the orthocentre, the in-centre, the circum-centre and ex-centres. Remarkable relations, which are connected with these six points (in the general case), are the Euler line, the Euler circle of the ΔABC and the isogonal conjugates with respect to ΔABC. It is curious to examine whether in substituting any of the mentioned points by arbitrary one from the plane of the ΔABC (with possible exceptions of natural character), the other five points are determined uniquely, thus transforming the known lines and circles into generalizations with similar properties. We propose an analysis of some characteristic properties of the already mentioned notable points of the triangle by an essential assistance of the GSP, and we show the existence of corresponding generalizations, which are analogous to already known figures from the geometry of the triangle.

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In such a way we demonstrate the joint work of the analogy and the computer software GSP to notice generalizations of well-known theorems from the geometry of the triangle. By the help of the GSP we examine the dependence of some triangle properties on an arbitrary point. The whole process should be considered as finalized only in case the results are proved in a strictly mathematical way. The proofs are elaborated using barycentric coordinates with respect to the given triangle ABC, namely $A(1,0,0)$, $B(0,1,0)$ and $C(0,0,1)$ [1]. The midpoints of the sides BC, CA and AB are denoted by $M_a(0,\frac{1}{2},\frac{1}{2})$, $M_b(\frac{1}{2},0,\frac{1}{2})$ and $M_c(\frac{1}{2},\frac{1}{2},0)$, respectively.

2. EULER LINE AND EULER CURVE, DEPENDING ON A POINT

Every non-equilateral triangle has a special line, known to be Euler line and also a special circle – Euler circle [1], [2]. Following reasoning by analogy we will show by means of the computer program GSP how a similar line and a curve of second degree for a given triangle could be obtained in dependence on an arbitrary point in the plane of the triangle.

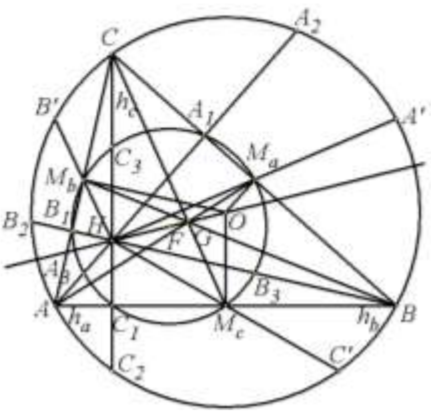

Figure 1

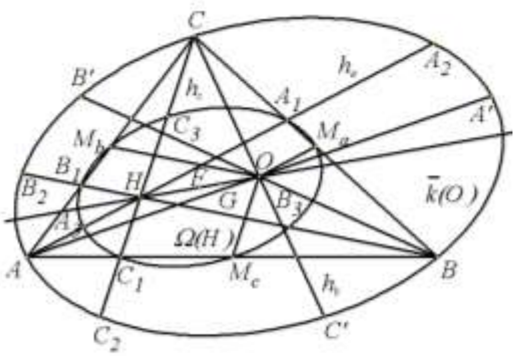

Figure 2

In the general case, characteristic points of ΔABC on the Euler line are the circum-centre, the orthocentre, the gravity centre and the centre of the Euler circle (Fig. 1). The circum-circle of the ΔABC is one element only of an infinity set of second degree curves, which are circumscribed round ΔABC. Something more, if $O(x_0,y_0,z_0)$ $(x_0+y_0+z_0=1)$ is an arbitrary point, which is not concurrent with the lines BC, CA, AB, M_bM_c, M_cM_a and M_aM_b in the plane of the ΔABC, then A, B, C and correspondingly their symmetric points with respect to O

$$A'(2x_0-1,2y_0,2z_0),\ B'(2x_0,2y_0-1,2z_0),\ C'(2x_0,2y_0,2z_0-1)$$

lie on a curve of second degree $\bar{k}(O)$ with centre O (Fig. 2, 3). The curve $\bar{k}(O)$ has the following equation:

$$\bar{k}(O):(1-2x_0)x_0yz+(1-2y_0)y_0zx+(1-2z_0)z_0xy=0\ . \qquad (1)$$

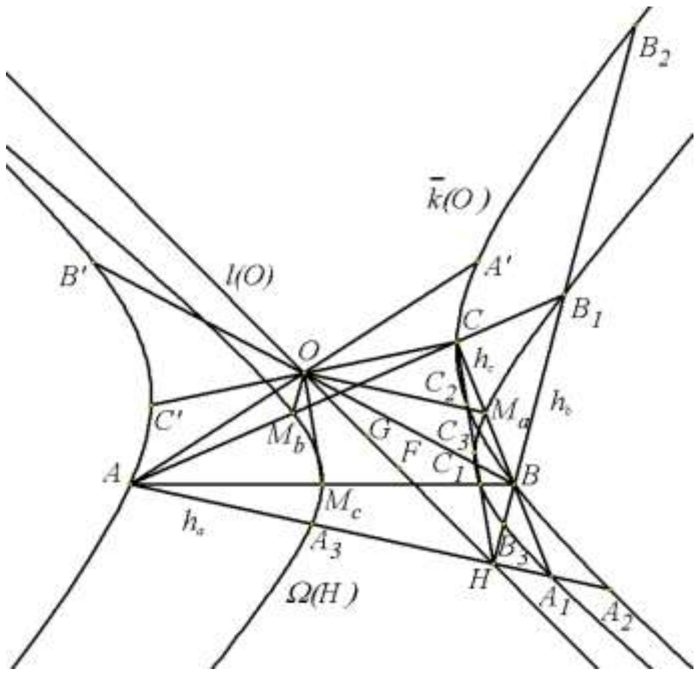

Figure 3

The point O, thus determined is analogous to the circum-centre of ΔABC. Now, we will define a point, which is analogous to the orthocentre of ΔABC. The altitudes of ΔABC are parallel to the lines, which pass through the circum-centre and the points M_a, M_b and M_c (Fig. 1). This is a reason to construct the lines h_a, h_b and h_c by the GSP, passing through the vertices A, B, C, respectively and parallel to the lines OM_a, OM_b and OM_c, respectively. It is seen that these lines have a common point H (Fig. 2, 3). Additionally, no matter how the position of O is changed, the lines under consideration have a common point always. In such a way the following assertion is obtained:

Property 1. *The lines* h_a , h_b *and* h_c *have a common point* H .

For the proof we find parametric equations of the lines h_a , h_b and h_c :

$$h_a : x = 1 + x_0 t_a, y = (y_0 - \tfrac{1}{2}) t_a, z = (z_0 - \tfrac{1}{2}) t_a,$$
$$h_b : x = (x_0 - \tfrac{1}{2}) t_b, y = 1 + y_0 t_b, z = (z_0 - \tfrac{1}{2}) t_b,$$
$$h_c : x = (x_0 - \tfrac{1}{2}) t_c, y = (y_0 - \tfrac{1}{2}) t_c, z = 1 + z_0 t_c.$$

Further, we solve the system with the equations of the lines h_a and h_b. We find that the common point is $H(1-2x_0, 1-2y_0, 1-2z_0)$. Finally, we check that the coordinates of this point satisfy the equations of h_c .

For a more convincing analogy between the point H and the orthocentre of ΔABC, this point should possess other properties, which are characteristic for the orthocentre. Let the lines h_a , h_b and h_c intersect the lines BC , CA and AB in the points A_1, B_1 and C_1, respectively, and also the curve $\overline{k}(O)$ – in the points A_2, B_2 and C_2, respectively (Fig. 2, 3). It is well-known that the points, which are symmetric to the orthocentre with respect to the sides and the midpoints of the sides, lie of the circum-circle of ΔABC (Fig. 1) [1]. For this reason we check by the GSP whether the points A_2, B_2 and C_2 are symmetric to H with respect to A_1, B_1 and C_1, correspondingly. We check also whether the points A', B' and C' are symmetric to H with respect to M_a, M_b and M_c, correspondingly. All constructions lead to the conclusion that H has the following properties:

Property 2. *The points* A_2, B_2 *and* C_2 *are symmetric to* H *with respect to* A_1, B_1 *and* C_1, *correspondingly.*

Property 3. *The points* A', B' *and* C' *are symmetric to* H *with respect to* M_a, M_b *and* M_c, *correspondingly.*

The proofs of these properties could be obtained using the coordinates of the coresponding ponts.

Because of the shown analogy of the constructed point H with the orthocentre of ΔABC, we will call this point to be o*rthoid of* ΔABC, *depending on the point* O .

In the GSP let us construct now the line OH and the gravity centre $G(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ of ΔABC. We establish, that:

Property 4. *The points O, H and G are collinear and the point G divides the segment HO in ratio $2:1$.*

The proof ot this property could be obtained by a reasoning, that the coordinates of the points O, H and G imply the vector equality $\overrightarrow{HG} = 2.\overrightarrow{GO}$. The property 4 shows, that the line $l(O)$, containing the points O, H and G is an analogue to the Euler line and for this reason we will say, that $l(O)$ is *Euler line of ΔABC, depending on the point O.*

The centre of the Euler circle of ΔABC lies on the Euler line too. Let us look for a curve, which is analogous to the Euler circle. In the general case the Euler circle contains the points M_a, M_b and M_c, the feet of the altitudes and the midpoints of the segments connecting the vertices of the triangle with its orthocentre (Fig. 1) [1].

For an arbitrary point $P(\lambda,\mu,\nu)$ $(\lambda+\mu+\nu=1)$ from the plane of ΔABC, the points M_a, M_b and M_c, the common points of the lines AP, BP and CP, denoted with A_3, B_3 and C_3, respectively, lie on a curve of second degree $\Omega(P)$ (Fig. 2, 3). We call it to be *Euler curve for the point P with respect to ΔABC*. This curve has the following equation:

$$\Omega(P): \mu\nu x^2 + \nu\lambda y^2 + \lambda\mu z^2 - (1-\lambda)\lambda yz - (1-\mu)\mu zx - (1-\nu)\nu xy = 0 .$$

Obviously, the Euler circle of ΔABC is Euler curve of its orthocentre. Since H is an analogue to the orthocentre, it is reasonable in this case to consider the Euler curve $\Omega(H)$ of the point H. It follows from the definition, that $\Omega(H)$ passes through the points M_a, M_b, M_c, A_1, B_1, C_1 and the midpoints of the segments AH, BH and CH, and its equation is

$$(1-2y_0)(1-2z_0)x^2 + (1-2z_0)(1-2x_0)y^2 + (1-2x_0)(1-2y_0)z^2 -$$
$$-2(1-2x_0)x_0 yz - 2(1-2y_0)y_0 zx - 2(1-2z_0)z_0 xy = 0.$$

We say, that $\Omega(H)$ *is Euler curve of ΔABC, depending on the point O.* Experimenting by the GSP we establish for the points A_3, B_3 and C_3, that

Property 5. *The lines $A'A_3$, $B'B_3$ and $C'C_3$, pass through the point G and the relations $GA' : GA_3 = GB' : GB_3 = GC' : GC_3 = 2:1$ are satisfied.*

The coordinates of the points under consideration imply the vector equalities $\overrightarrow{GA_3} = -\frac{1}{2}\overrightarrow{GA'}$, $\overrightarrow{GB_3} = -\frac{1}{2}\overrightarrow{GB'}$ and $\overrightarrow{GC_3} = -\frac{1}{2}\overrightarrow{GC'}$, which prove the property.

Now, let $h(H,\frac{1}{2})$ be a homothety with centre H and coefficient $\frac{1}{2}$, while $h(G,-\frac{1}{2})$ be a homothety with centre O and $-\frac{1}{2}$. We obtain from properties 2, 3, 5 and the main property of the gravity centre, that:

Property 6. *The homotheties* $h(H,\frac{1}{2})$ *and* $h(G,-\frac{1}{2})$ *transform* $\overline{k}(O)$ *to* $\Omega(H)$.

It follows directly from the last property, that

Property 7. *The curves* $\overline{k}(O)$ *and* $\Omega(H)$ *are of the same type.*

Let $F(\frac{1-x_0}{2},\frac{1-y_0}{2},\frac{1-z_0}{2})$ be the midpoint of the segment OH. It follows from property 6, that

Property 8. *The point* F *is centre of the Euler curve* $\Omega\left(H\right)$.

We have also the following

Property 9. *The homotheties* $h(H,\frac{1}{2})$ *and* $h(G,-\frac{1}{2})$ *transform all points from* $\overline{k}(O)$ *to diametrically opposite points from* $\Omega(H)$.

The obtained results imply, that the line $l\left(O\right)$ does not exist exactly when $O \equiv G$. In this case $\Omega(G)$ is an ellipse, inscribed in ΔABC. This fact explains why the equilateral triangle has no Euler line.

Up to now, we have considered the cases, when the circum-curve of ΔABC is an ellipse or a hyperbola with center in a given point O. The circum parabolas of ΔABC could be considered as conic sections with infinity centers [3]. The infinity center O of the parabola could be determined by the directrix of a given vector $\vec{O}$. Let $\vec{O}(x_0, y_0, z_0)$ $(x_0 + y_0 + z_0 = 0)$ be a vector, which is not collinear with any of the lines BC, CA and AB. A unique parabola $k(\vec{O})$ exists, which passes through the points A, B and C. It has an axis, which is a collinear line with $\vec{O}$ (it touches the infinity line of the plane in the infinity point $\vec{O}$) [3]. The parabola $k(\vec{O})$ has the following equation:

$$k(\vec{O}): x_0^2 yz + y_0^2 zx + z_0^2 xy = 0 .$$

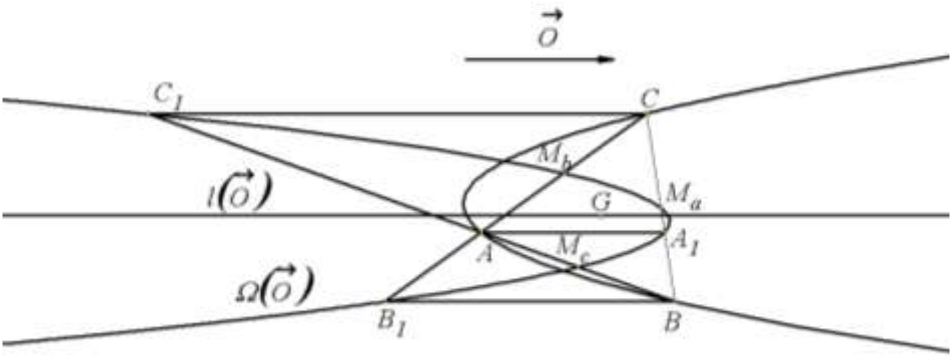

Figure 4

In this case the point H could be considered as coinciding with the infinity center of $k(\vec{O})$. We consider the line $l(\vec{O})$ through G and collinear with $\vec{O}$ to be *Euler line of* ΔABC, *depending on the directrix* $\vec{O}$ *(or, which is the same, depending on the infinity point* $\vec{O}$, *denoting it by* $l(\vec{O})$).

We construct the lines a_0, b_0 and c_0, which pass through the vertices A, B and C, correspondingly and are collinear with the vector $\vec{O}$. Let $a_0 \cap BC = A_1$, $b_0 \cap CA = B_1$ and $c_0 \cap AB = C_1$. The coordinates of the points M_a, M_b, M_c, A_1, B_1, C_1 and the vector $\vec{O}$ satisfy the equation

$$\Omega(\vec{O}): y_0 z_0 x^2 + z_0 x_0 y^2 + x_0 y_0 z^2 + x_0^2 yz + y_0^2 zx + z_0^2 xy = 0 .$$

The parabola $\Omega(\vec{O})$ is considered to be *Euler curve of* ΔABC, *depending on the direcrix* $\vec{O}$ *(or, which is the same, depending on the infinity point).*

The parabolas $k(\vec{O})$ and $\Omega(\vec{O})$ are connected with the following

Property 10. *The homothety* $h(G,-\frac{1}{2})$ *transforms the parabola* $\Omega(\vec{O})$ *into the parabola* $k(\vec{O})$.

The last property shows, that the circum parabolas of ΔABC look like the circum ellipse of ΔABC with center in the point G because in each of the two cases there is only one homothety, transforming the circum curve into the corresponding Euler curve.

Notice, that the infinity points also generate Euler lines and Euler curves (parabolas) of the given triangle. However, in this case the properties are less abundant because they are connected with circum parabolas round the triangle. The parabolas are placed more especially in the set of the three conic sections with respect to the centre notion. For this reason they transfer the peculiarity to the questions under consideration.

The above investigations show, that Euler line $l(O)$ and Euler curve $\Omega(H)$ could be associated to each finite or infinite point O from the plane of a given triangle ABC, which is not concurrent with any of the lines BC, CA, AB, M_bM_c, M_cM_a, M_aM_b and which is different from the gravity center G. The line $l(O)$ and the curve $\Omega(H)$ coincide with the classic ones exactly when O is the circumcenter of a non-equilateral triangle ΔABC.

3. Generalization of the Feuerbach theorem

A well-known Feuerbach theorem from the geometry of the triangle asserts, that the Euler circle of a triangle is tangent to the four in-circles of this triangle [2], [4]. What follows in the sequel is a generalization of this theorem.

By analogous reasoning, notifying some main dependences among the circles from the Feuerbach theorem, we will show how to obtain a generalization of this theorem by the computer program GSP, in which the participation of circles is not obligatory. Firstly, we will notify the generation of a triangle, which is conjugated to arbitrary point P with respect to the given ΔABC.

Let the points A_1, B_1 and C_1 be on the lines BC, CA and AB, respectively and be such, that the lines AA_1, BB_1 and CC_1, pass through a point P (finite or infinite). If the points A_2, B_2 and C_2 are the harmonic conjugated to A_1, B_1 and C_1, respectively, with respect to the couples of vertices of the given triangle, then the lines AA_2, BB_2 and CC_2 are called to be harmonic conjugated to the lines AP, BP and CP with respect to ΔABC [3]. Let $BB_2 \cap CC_2 = P_a$, $CC_2 \cap AA_2 = P_b$ and $AA_2 \cap BB_2 = P_c$, then the triangle $\Delta P_aP_bP_c$ is called to be conjugated (harmonic) to the point P with respect to

ΔABC [1]. It is easy to show, that the conjugated triangle of P_a is $\Delta P_b P_c P$, of P_b is $\Delta P_c P_a P$, and of P_c is $\Delta P_a P_b P$ [1]. For this reason we will say for such four points in the plane of ΔABC, that they form a harmonic four of points with respect to ΔABC.

The first thing, on which we should pay attention in the Feuerbach theorem, is that five circles participate in it and four of them are tangent to the lines BC, CA and AB. Since the circle is a special type ellipse and the ellipse is a type central conic section, then the desired generalization should be in search through the inscribed conic sections of the given triangle. At the beginning the generalization will be in searching through ellipses and hyperbolas, after that through the inscribed parabolas.

The next thing, on which we should pay attention, is that the four in-circles participate in the Feuerbach theorem jointly and not separately — each for itself. Consequently, their centers participate jointly too. A remarkable property of the four centers is that they form a harmonic four of points with respect to ΔABC.

Construct a triangle ABC in the GSP program and also an arbitrary harmonic four I, I_A, I_B and I_C. Let $I(x_I, y_I, z_I)$, $I_A(-\frac{x_I}{1-2x_I}, \frac{y_I}{1-2x_I}, \frac{z_I}{1-2x_I})$, $I_B(\frac{x_I}{1-2x_I}, -\frac{y_I}{1-2x_I}, \frac{z_I}{1-2x_I})$ and $I_C(\frac{x_I}{1-2x_I}, \frac{y_I}{1-2x_I}, -\frac{z_I}{1-2x_I})$ [1] be points, which form a harmonic four of points with respect to ΔABC.

Another remarkable property of the in-centers is that they define segments, whose midpoints lie on the circum-circle of ΔABC. For this reason we consider the midpoints M_A, M_B, M_C, M_{BC}, M_{CA} and M_{AB} of the segments of II_A, II_B, II_C, $I_B I_C$, $I_C I_A$ and $I_A I_B$, respectively. Now, we should expect, that the points M_A, M_B, M_C, M_{BC}, M_{CA} and M_{AB} lie on a central conic section, circumscribed round ΔABC. The case when the points I, I_A, I_B and I_C are the centers of the in-circles of ΔABC, the points M_A, M_B, M_C, M_{BC}, M_{CA} and M_{AB} are midpoints of arcs on the circum-circle of ΔABC. The perpendicular bisectors of the segments BC, CA and AB pass through the same midpoints too. On the other hand, these perpendicular bisectors pass also through the midpoints M_a, M_b, M_c of the segments BC, CA and AB, respectively. Therefore, we could expect, that in this case the lines $M_A M_a$, $M_B M_b$ and $M_C M_c$ are analogous to the perpendicular bisectors and for this reason they should have a common point. Construct the lines $M_A M_a$, $M_B M_b$

and $M_C M_c$ in the computer program GSP. The construction shows, that these lines pass through a point $O(x_0, y_0, z_0)$ $(x_0 + y_0 + z_0 = 1)$ and $M_{BC} \in M_A M_a$, $M_{CA} \in M_B M_b$ and $M_{AB} \in M_C M_c$, which visualizes the correctness of the supposition. The analytic computations in [9] confirm the observed concurrence. Additionally, it is established that the points M_A, M_B, M_C, M_{BC}, M_{CA} and M_{AB} lie on the circumscribed conic section $\bar{k}(O)$ of ΔABC with equation (1), while the coordinates of its center are determined by the equalities

$$x_0 = \frac{(1-2x_I-2y_I z_I)x_I^2}{(1-2x_I)(1-2y_I)(1-2z_I)},$$

$$y_0 = \frac{(1-2y_I-2z_I x_I)y_I^2}{(1-2x_I)(1-2y_I)(1-2z_I)},$$

$$z_0 = \frac{(1-2z_I-2x_I y_I)z_I^2}{(1-2x_I)(1-2y_I)(1-2z_I)}.$$

Another property of the in-centers is that the lines determined by the centers and the tangent points are perpendicular to the lines BC, CA and AB. For this reason these lines are parallel to the perpendicular bisectors of the segments BC, CA and AB. Use the notations $k(I)$, $k(I_A)$, $k(I_B)$ and $k(I_C)$ for the inscribed conic sections of ΔABC, whose centers are the points I, I_A, I_B and I_C, respectively. Further we will examine $k(I)$ only, because the remaining inscribed conic sections have the same properties.

Use the notations A_I, B_I and C_I for the tangent points of $k(I)$ with the lines BC, CA and AB, respectively. It follows from the already done observations, that the lines IA_I, IB_I and IC_I should be parallel to the lines OM_a, OM_b and OM_c, respectively. Thus, using the GSP program we construct the lines IA_I, IB_I and IC_I through the point I, which are parallel to the lines OM_a, OM_b and OM_c, respectively. Further, we construct the points A_I, B_I and C_I, as intersections of the lines IA_I, IB_I and IC_I, with the lines BC, CA and AB, respectively and also the curve $k(I)$, which passes through these points and whose center is the point I. We see, that $k(I)$ has no other common point with the lines BC, CA and AB. Consequently, $k(I)$ is an inscribed curve of ΔABC. The observations are proved strictly in [5], where it is shown, that the points A_I, B_I and C_I lie on the in-curve $k(I)$ of ΔABC, whose equation is the following:

$$(1-2x_I)^2x^2+(1-2y_I)^2y^2+(1-2z_I)^2z^2-2(1-2y_I)(1-2z_I)yz-$$
$$-2(1-2z_I)(1-2x_I)zx-2(1-2x_I)(1-2y_I)xy=0.$$

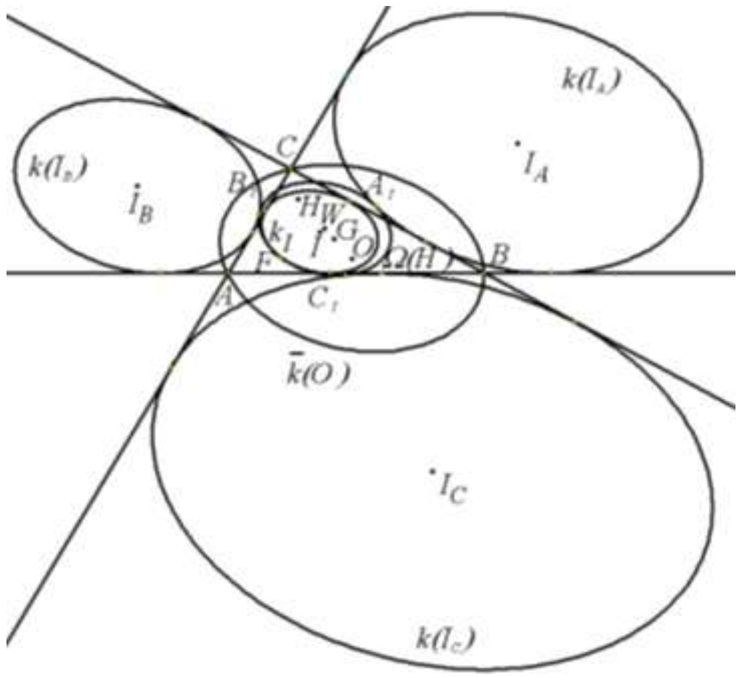

Figure 5

Let H be the point depending on O by which O determines the Euler line $l(O)$. Construct the Euler curve $\Omega(H)$ for the point H using the computer program GSP. Notice, that this curve is tangent to $k(I)$, $k(I_A)$, $k(I_B)$ and $k(I_C)$ consequently, this is the generalization of the Feuerbach theorem in search. Thus, it is confirmed one more time again, that $\Omega(H)$ is the true analogue to the Euler circle, which depends on a point from the plane of ΔABC . After using the equations of the curves $k(I)$, $\Omega(H)$ and also the dependence among the coordinates of the centers O and I, it is shown in в [5], that $k\left(I\right)$ and $\Omega(H)$ are tangent in the point

$$F\left(\frac{(2x_I-1)(y_I-z_I)^2}{f},\frac{(2y_I-1)(z_I-x_I)^2}{f},\frac{(2z_I-1)(x_I-y_I)^2}{f}\right),$$

where $f = 2((1-2x_I)(1-2y_I)(1-2z_I) - x_I y_I z_I)$.

In such a way, the following generalization of the Feuerbach theorem is established:

Theorem 1. *The inscribed conic sections* $k(I)$, $k(I_A)$, $k(I_B)$ *and* $k(I_C)$ *of* ΔABC *are tangent to the Euler curve* $\Omega(H)$ (Fig. 5).

It is seen from the constructions, already performed in the computer program GSP, that the five curves are ellipses or hyperbolas simultaneously. Something more, when they are hyperbolas it seems that they have the same asymptotic directions. It could be said in fact, that the five hyperbolas pass through two infinite points. The proof of these facts is performed in [5].

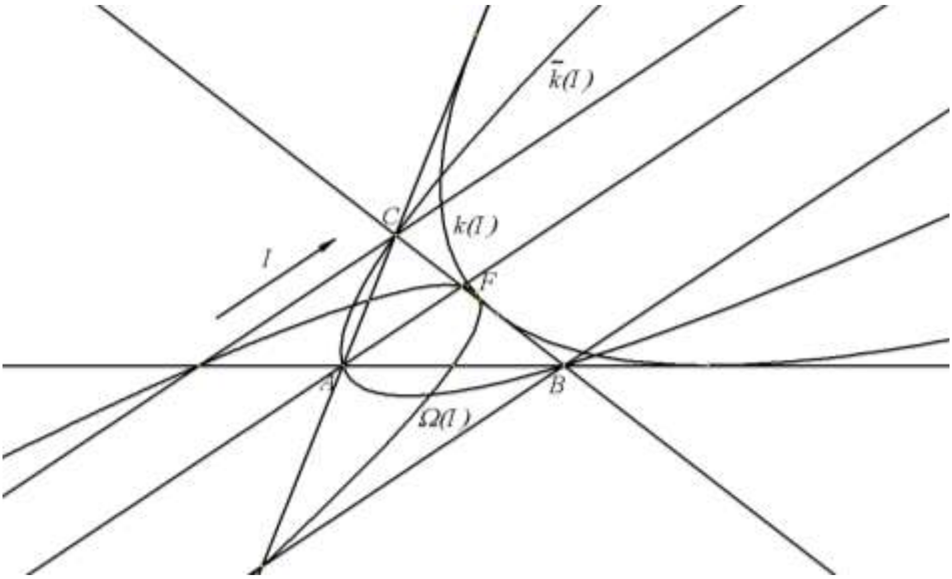

Figure 6

Now let $I(x_I, y_I, z_I)$ be infinite point, i.e. $x_I + y_I + z_I = 0$. The points $I_A(\frac{1}{2}, -\frac{y_I}{2x_I}, -\frac{z_I}{2x_I})$, $I_B(-\frac{x_I}{2y_I}, \frac{1}{2}, -\frac{z_I}{2y_I})$ and $I_C(-\frac{x_I}{2z_I}, -\frac{y_I}{2z_I}, \frac{1}{2})$ [5] together with the point I form a harmonic four with respect to ΔABC. These points are finite and they lie on the lines M_bM_c, M_cM_a, M_aM_b. As stated in [5], the points from these lines could not be centers of conic sections, inscribed in

ΔABC. Consequently, the point I generates only one conic section, inscribed in ΔABC.

We consider the parabola $k(I)$ as a conic section with infinite center I. There exists exactly one parabola $k(I)$, which has an infinite center I and is tangent to the lines BC, CA and AB. The equation of the parabola $k(I)$ is $x_I^2x^2 + y_I^2y^2 + z_I^2z^2 - 2y_Iz_Iyz - 2z_Ix_Izx - 2x_Iy_Ixy = 0$, which together with the equation of the Euler parabola $\Omega(I)$ results in a unique common point for the two curves. It has the following coordinates:

$$F(-\frac{(y_I-z_I)^2}{9y_Iz_I}, -\frac{(z_I-x_I)^2}{9z_Ix_I}, -\frac{(x_I-y_I)^2}{9x_Iy_I}).$$

Thus, we establish

Theorem 2. *If I is an arbitrary infinite point from the plane of ΔABC, then the generated by this point inscribed parabola and Euler curve $\Omega(I)$ of ΔABC are tangent* (Fig. 6).

The case with the parabola looks like the case with hyperbola regarding the common infinite point of the inscribed parabola and the Euler curve (parabola). In such a way it turns out, that the two parabolas have two tangent points — one finite and one infinite.

The above reasoning shows, that in each concrete case all curves under consideration are of the same type.

Some interesting properties of examined constructions are contain in [6] and [7].

4. CONCLUSIONS

The challenge to find a generalization of a geometric theorem is connected with a deep understanding of the considered figure properties. A necessary step is to clarify the relation among the elements of a given configuration, thus extracting the properties which could be changed. How to perform the change? Which elements and properties should be modified in order to change the corresponding theorem itself? The GSP program turns out to be useful instrument in the process of answering these questions. The theorems included in the paper are mostly from the triangle geometry and are connected with different classes of circles, lines and points in the plane of that triangle. After the analysis of the corresponding relations the circles, the lines and the points are replaced by suitable conics, lines and points thus keeping the validity of the theorems in the new situation. A deep knowledge of conic properties and constructive skills are necessary for the purpose. The program GSP is applied for fast elimination of various conjectures which turn out to be false, but also for the creation of convincing configurations leading to the formulation of the desired assertions. The assertions themselves should be considered as true only in case they are strictly proven. Generalizations are obtained in many cases but reasons are found very often to reject some.

The established generalizations propose a new view on well-known geometric theorems and expose deeper sense of the participating figures. They give possibilities to overcome the limits of previous perceptions. Thus, a gradual deepening of the understanding concerning projective properties of conics is realized. Experience is obtained to discover certain theorems, which helps further investigations making them easier.

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