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REMARK ABOUT CHARACTERIZATION OF 2-INNER PRODUCT

Katerina Anevska[1] and Risto Malčeski[2]

Abstract. Characterization of 2-inner product is focus of interest of many mathematicians. In this paper proofs of two characterizations of 2- inner product, which are actually consequences of the Theorem 1 [15], are given. Also, generalizations of already know Hayashi (see [4], pg. 297) and Zarantonello ([5]) inequalities are fully elaborated.

1. INTRODUCTION

The concepts of 2-norm and 2-inner product are two-dimensional analogies to the concepts of norm and inner product. S. Gähler ([13]), 1965, gave the term of 2-norm and R. Ehret ([11]), 1969, proved the following:

If $(L,(\cdot,\cdot|\cdot))$ is a 2-pre-Hilbert space, then

$$\|x,y\|=(x,x|y)^{1/2}, \tag{1}$$

for all $x,y\in L$, defines a 2-norm. So, we get the 2-normed space $(L,\|\cdot,\cdot\|)$ and furthermore for all x, y, z ϵ L the following equalities are satisfied:

$$(x,y|z)=\frac{\|x+y,z\|^2-\|x-y,z\|^2}{4}, \tag{2}$$

$$\|x+y,z\|^2+\|x-y,z\|^2=2(\|x,z\|^2+\|y,z\|^2), \tag{3}$$

The equality (3) is analogue to the parallelogram equality, and it is said to be parallelepiped equality. Moreover, 2-normed space L is 2-pre-Hilbert if and only if the equality (3) is satisfied for all $x,y,z\in L$.

The papers [1]-[3], [6], [12], [14]-[16] consist of many proven characterizations about 2-inner product.

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Theorem 1 ([15]). Let $(L,\|\cdot,\cdot\|)$ be a real 2-normed space. Then, L is a 2-pre-Hilbert space if and only if the following condition is satisfied:
if $n \geq 3$, $x_1, x_2, ..., x_n, z \in L$ and $a_1, a_2, ..., a_n$ are real numbers such that $\sum_{i=1}^{n} a_i = 0$, then

$$\| \sum_{i=1}^{n} a_i x_i, z \|^2 = -\sum_{1\leq i<j\leq n} a_i a_j \| x_i - x_j, z \|^2 \text{ . } \blacksquare \tag{4}$$

2. CHARACTERIZATION OF 2-PRE-HILBERT SPACE

The characterization of 2-inneer product by applying the Euler-Lagrange type of equality is elaborated in [6] or in other words generalization of Corollary 2.2 [8], is elaborated. The following theorem is one other proof of the above stated generalization.

Theorem 2 ([6]). Let $(L,\|\cdot,\cdot\|)$ be a real 2-normed space. The 2-norm is generated by 2-inner product if and only if the following equality is satisfied

$$\frac{\|ax+by,z\|^2}{\gamma} + \frac{\|\beta bx-\alpha ay,z\|^2}{\gamma\alpha\beta} = \frac{\|x,z\|^2}{\alpha} + \frac{\|y,z\|^2}{\beta}, \tag{5}$$

for all $x, y, z \in L$ and for all $a,b \in \mathbf{R}$, $\alpha, \beta > 0, \gamma = \alpha a^2 + \beta b^2$.

Proof. Let L be a real 2-normed space such that for all $x, y, z \in L$ and for all $a,b \in \mathbf{R}$, $\alpha, \beta > 0, \gamma = \alpha a^2 + \beta b^2$ the equality (5) is satisfied. For $\alpha = \beta = a = b = 1$, the equality (5) is transformed to an equality which is equivalent to the parallelepiped equality, (3), what actually means that L is 2-pre-Hilbert space in which the 2-inner product is defined as (2) and moreover (1) holds true.

Conversely, let 2-inner product, which determines the 2-norm, exist and let $x, y, z \in L$ and $a,b \in \mathbf{R}$, $\alpha, \beta > 0$ be such that $\gamma = \alpha a^2 + \beta b^2$ is satisfied. For

$$a_1 = \frac{a}{\sqrt{\gamma}}, a_2 = \frac{b}{\sqrt{\gamma}}, a_3 = -\frac{a+b}{\sqrt{\gamma}}, x_1 = x, x_2 = y, x_3 = 0$$

theorem 1 is transformed as the following

$$\frac{\|ax+by,z\|^2}{\gamma} = \frac{a(a+b)}{\gamma} \| x, z \|^2 + \frac{b(a+b)}{\gamma} \| y, z \|^2 - \frac{ab}{\gamma} \| x - y, z \|^2 . \tag{6}$$

Further, for

$$a_1=\frac{b\sqrt{\beta}}{\sqrt{\alpha\gamma}},\ a_2=-\frac{a\sqrt{\alpha}}{\sqrt{\beta\gamma}},\ a_3=-\frac{b\beta-a\alpha}{\sqrt{\alpha\beta\gamma}},\ x_1=x,\ \ x_2=y,\ x_3=0$$

theorem 1 is transformed as the following

$$\frac{\|\beta bx-\alpha ay,z\|^2}{\gamma\alpha\beta}=\frac{b(b\beta-a\alpha)}{\alpha\gamma}\|x,z\|^2-\frac{a(b\beta-a\alpha)}{\beta\gamma}\|y,z\|^2+\frac{ab}{\gamma}\|x-y,z\|^2. \qquad (7)$$

Finally, if we summarize the equalities (6) and (7) and have also on mind that $\gamma=\alpha a^2+\beta b^2$ we get the following

$$\begin{aligned}\frac{\|ax+by,z\|^2}{\gamma}+\frac{\|\beta bx-\alpha ay,z\|^2}{\gamma\alpha\beta}&=\left(\frac{a(a+b)}{\gamma}+\frac{b(b\beta-a\alpha)}{\alpha\gamma}\right)\|x,z\|^2+\\&\quad+\left(\frac{b(a+b)}{\gamma}-\frac{a(b\beta-a\alpha)}{\beta\gamma}\right)\|y,z\|^2\\&=\frac{\alpha a^2+\beta b^2}{\alpha\gamma}\|x,z\|^2+\frac{\alpha a^2+\beta b^2}{\beta\gamma}\|y,z\|^2\\&=\frac{\|x,z\|^2}{\alpha}+\frac{\|y,z\|^2}{\beta},\end{aligned}$$

i.e. the equality (5) is satisfied. ■

The following theorem is actually generalization of M. S. Moslehian and J. M. Rassias (Corollary 2.2, [9]) result.

Theorem 3.A real 2- normed space $(L,\|\cdot,\cdot\|)$ is 2- pre-Hilfert space if and only if for each $n\geq 2$ and for all $x_1,x_2,...,x_n,z\in L$ the equality (8) is satisfied

$$\sum_{a_i\in\{-1,1\}}\|x_1+\sum_{i=2}^{n}a_ix_i,z\|^2=\sum_{a_i\in\{-1,1\}}(\|x_1,z\|+\sum_{i=2}^{n}a_i\|x_i,z\|)^2. \qquad (8)$$

Proof. Let (8) be satisfied for each $n\geq 2$ and for all $x_1,x_2,...,x_n,z\in L$. For $n=2$, $x_1=x$ and $x_2=y$ the equality (8) is transformed to the parallelepiped equality (3). That actually means that L is 2-pre-Hilbert space in which the 2-inner product is defined as (2) and furthermore (1) holds true.

Conversely, let a 2-normed space $(L,\|\cdot,\cdot\|)$ be a 2-pre-Hilbert space, $n\geq 2$ and $x_1,x_2,...,x_n,z\in L$.

For $a_{n+1}=-(1+\sum_{k=2}^{n}a_i)$ and $x_{n+1}=0$, Theorem 1 is transformed as the following:

$$\| x_1 + \sum_{i=2}^{n} a_i x_i , z \|^2 = \| x_1 + \sum_{i=2}^{n+1} a_i x_i , z \|^2$$

$$= (1 + \sum_{k=2}^{n} a_k)(\| x_1, z \|^2 + \sum_{i=2}^{n} a_i \| x_i , z \|^2) -$$

$$- \sum_{i=2}^{n} a_i \| x_1 - x_i , z \|^2 - \sum_{2 \le i < j \le n} a_i a_j \| x_i - x_j , z \|^2$$

$$= \sum_{i=1}^{n} \| x_i , z \|^2 + \| x_1, z \|^2 \sum_{k=2}^{n} a_k + \sum_{k=2}^{n} \sum_{\substack{i=2 \\ i \ne k}}^{n} a_k a_i \| x_i , z \|^2 -$$

$$- \sum_{i=2}^{n} a_i \| x_1 - x_i , z \|^2 - \sum_{2 \le i < j \le n} a_i a_j \| x_i - x_j , z \|^2$$

and since $a_i \in \{-1,1\}$, for $i = 2,3,...,n$, we get 2^{n-1} equalities of the above type. By summarizing the such obtained equalities, we get the following.

$$\sum_{a_i \in \{-1,1\}} \| x_1 + \sum_{i=2}^{n} a_i x_i , z \|^2 = 2^{n-1} \sum_{i=1}^{n} \| x_i , z \|^2 + \sum_{a_k \in \{-1,1\}} \sum_{k=2}^{n} a_k \| x_1, z \|^2 +$$

$$+ \sum_{a_k, a_i \in \{-1,1\}} \sum_{k=2}^{n} \sum_{\substack{i=2 \\ i \ne k}}^{n} a_k a_i \| x_i , z \|^2 -$$

$$- \sum_{a_k, a_i \in \{-1,1\}} \sum_{i=2}^{n} a_i \| x_1 - x_i , z \|^2 -$$

$$- \sum_{a_k, a_i \in \{-1,1\}} \sum_{2 \le i < j \le n} a_i a_j \| x_i - x_j , z \|^2$$

$$= 2^{n-1} \sum_{i=1}^{n} \| x_i , z \|^2$$

$$= \sum_{a_i \in \{-1,1\}} (\| x_1, z \| + \sum_{i=2}^{n} a_i \| x_i , z \|)^2 ,$$

i.e. the equality (8) holds true. ■

3. Generalization Of Hayashi And Zarantonello Inequalities

The following theorems, are actually generalization of two already known equalities, obtained by using theorem 1. Thus, we will firstly give a generalization of Hayashi (see [4], pg. 297) inequality for complex numbers.

Theorem 4. Let $(L,\|\cdot,\cdot\|)$ be a real 2-normed space. Then

$$\sum_{cyclic}\|x-x_1,z\|\cdot\|x-x_2,z\|\cdot\|x_1-x_2,z\|\geq\|x_1-x_2,z\|\cdot\|x_2-x_3,z\|\cdot\|x_3-x_1,z\| \quad (9)$$

for all $x,x_1,x_2,x_3,z\in L$. The inequality is transformed to an equality, if at least one of the sets $\{x-x_1,z\},\{x-x_2,z\},\{x-x_3,z\}$ is linearly dependent or more over if the set

$$\{\tfrac{\|x_2-x_3,z\|}{\|x-x_1,z\|}(x-x_1)+\tfrac{\|x_3-x_1,z\|}{\|x-x_2,z\|}(x-x_2)+\tfrac{\|x_1-x_2,z\|}{\|x-x_3,z\|}(x-x_3),z\}$$

is linearly dependent.

Proof. Let at least one of the sets $\{x-x_1,z\},\{x-x_2,z\},\{x-x_3,z\}$ be linearly dependent. With no loose of the generality, let $\{x-x_1,z\}$ be such the set, i.e. $x=x_1+\alpha z$. Then, the properties of 2-norm imply the following

$$\begin{aligned}\sum_{cyclic}\|x-x_1,z\|\cdot\|x-x_2,z\|\cdot\|x_1-x_2,z\|&=\|x-x_2,z\|\cdot\|x-x_3,z\|\cdot\|x_2-x_3,z\|\\&=\|x_1+\alpha z-x_2,z\|\cdot\|x_1+\alpha z-x_3,z\|\cdot\|x_2-x_3,z\|\\&=\|x_1-x_2,z\|\cdot\|x_2-x_3,z\|\cdot\|x_3-x_1,z\|,\end{aligned}$$

The above means that (9) is an equality.

Let's suppose that the sets $\{x-x_1,z\},\{x-x_2,z\},\{x-x_3,z\}$ are linearly independent. For $a_4=-\sum_{i=1}^{3}a_i$ and $x_4=x$ in Theorem 1, we get that for all $x,x_1,x_2,x_3,z\in L$ and for all $a_1,a_2,a_3\in\mathbf{R}$ the equality

$$\|\sum_{i=1}^{3}a_ix_i-x\sum_{i=1}^{3}a_i,z\|^2=(\sum_{i=1}^{3}a_i)\cdot(\sum_{i=1}^{3}a_i\|x-x_i,z\|)-\sum_{1\leq i<j\leq 3}a_ia_j\|x_i-x_j,z\|^2$$

holds true.

The right side of the above equality is nonnegative. Therefore, for all $x,x_1,x_2,x_3,z\in L$ and for all $a_1,a_2,a_3\in\mathbf{R}$ the inequality (10) holds true

$$(\sum_{i=1}^{3}a_i)\cdot(\sum_{i=1}^{3}a_i\|x-x_i,z\|)\geq\sum_{1\leq i<j\leq 3}a_ia_j\|x_i-x_j,z\|^2. \quad (10)$$

For

$$a_1=\tfrac{\|x_2-x_3,z\|}{\|x-x_1,z\|},\ a_2=\tfrac{\|x_3-x_1,z\|}{\|x-x_2,z\|},\ a_3=\tfrac{\|x_1-x_2,z\|}{\|x-x_3,z\|}$$

the inequality (10) is transformed as the followings

$$\sum_{i\neq j\neq k\neq i}\tfrac{\|x_i-x_j,z\|}{\|x-x_k,z\|}\sum_{i\neq j\neq k\neq i}\|x_i-x_j,z\|\cdot\|x-x_k,z\|\geq\sum_{i\neq j\neq k\neq i}\tfrac{\|x_j-x_k,z\|}{\|x-x_i,z\|}\tfrac{\|x_k-x_i,z\|}{\|x-x_j,z\|}\|x_i-x_j,z\|^2$$

$$\sum_{i\neq j\neq k\neq i}\frac{\|x_i-x_j,z\|}{\|x-x_k,z\|}\sum_{i\neq j\neq k\neq i}\|x_i-x_j,z\|\cdot\|x-x_k,z\|\geq$$

$$\geq\frac{\|x_1-x_2,z\|\cdot\|x_2-x_3,z\|\cdot\|x_3-x_1,z\|}{\|x-x_1,z\|\cdot\|x-x_2,z\|\cdot\|x-x_3,z\|}\sum_{i\neq j\neq k\neq i}\|x_i-x_j,z\|\cdot\|x-x_k,z\|$$

$$\sum_{i\neq j\neq k\neq i}\frac{\|x_i-x_j,z\|}{\|x-x_k,z\|}\geq\frac{\|x_1-x_2,z\|\cdot\|x_2-x_3,z\|\cdot\|x_3-x_1,z\|}{\|x-x_1,z\|\cdot\|x-x_2,z\|\cdot\|x-x_3,z\|}.$$

Clearly, the last inequality is equivalent to the inequality (9). The proof implies that the inequality (9) might be transformed to an equality if (10) is an equality, i.e. if the set $\{\sum_{i=1}^{3}a_ix_i-x\sum_{i=1}^{3}a_i,z\}$ is linearly dependent, that is if the set

$$\{\frac{\|x_2-x_3,z\|}{\|x-x_1,z\|}(x-x_1)+\frac{\|x_3-x_1,z\|}{\|x-x_2,z\|}(x-x_2)+\frac{\|x_1-x_2,z\|}{\|x-x_3,z\|}(x-x_3),z\}$$

is linearly dependent. ■

On the end of our considerations we will generalize the Zarantonello ([5]), inequality, i.e. we will prove the following theorem.

Theorem 5. Let L be a real 2-pre-Hilbert space and $f:L\to L$ be a function such that

$$\|f(x)-f(y),z\|\leq\|x-y,z\|, \tag{11}$$

holds true, for all $x,y,x\in L$, Then for all $a_1,a_2,...,a_n\geq 0$, such that $\sum_{i=1}^{n}a_i=1$ and for all $y_1,y_2,...,y_n,z\in L$

$$\|\sum_{i=1}^{n}a_if(y_i)-f(\sum_{k=1}^{n}a_ky_k),z\|^2\leq\sum_{1\leq i<k\leq n}a_ia_k(\|y_i-y_k,z\|^2-\|f(y_i)-f(y_k),z\|^2) \tag{12}$$

holds true.

Proof. For

$$x_i=f(y_i), i=1,2,..,.n,\ x_{n+1}=f(\sum_{i=1}^{n}a_iy_i)$$

and $a_{n+1}=-1$, in Theorem 1 and then by using the inequality (11) and the properties of 2-norm, we get that for all $a_1,a_2,...,a_n\geq 0$ such that $\sum_{i=1}^{n}a_i=1$ and for all $y_1,y_2,...,y_n,\ z\in L$, the following holds true

$$\|\sum_{i=1}^{n} a_i f(y_i) - f(\sum_{k=1}^{n} a_k y_k), z\|^2 =$$

$$= \sum_{i=1}^{n} a_i \|f(y_i) - f(\sum_{k=1}^{n} a_k y_k), z\|^2 - \sum_{1\le i<k\le n} a_i a_k \|f(y_i) - f(y_k), z\|^2$$

$$\le \sum_{i=1}^{n} a_i \|y_i - \sum_{k=1}^{n} a_k y_k, z\|^2 - \sum_{1\le i<k\le n} a_i a_k \|f(y_i) - f(y_k), z\|^2$$

$$= \sum_{i=1}^{n} a_i \|\sum_{k=1}^{n} a_k y_i - \sum_{k=1}^{n} a_k y_k, z\|^2 - \sum_{1\le i<k\le n} a_i a_k \|f(y_i) - f(y_k), z\|^2$$

$$= \sum_{i=1}^{n} a_i \|\sum_{k=1}^{n} a_k (y_i - y_k), z\|^2 - \sum_{1\le i<k\le n} a_i a_k \|f(y_i) - f(y_k), z\|^2.$$

On the other hand, for $x_k = y_i - y_k$, $k = 1,2,...,n$, $x_{n+1} = 0$ and $a_{n+1} = -1$ in Theorem 1 and also by using that $a_i \ge 0$, for $i = 1,2,...,n$, we get that

$$\|\sum_{k=1}^{n} a_k (y_i - y_k), z\|^2 = \sum_{k=1}^{n} a_k \|y_i - y_k, z\|^2 - \sum_{1\le i<j\le n} a_i a_j \|y_i - y_j, z\|^2$$

$$\le \sum_{k=1}^{n} a_k \|y_i - y_k, z\|^2 .$$

Finally, the last two inequalities imply the inequality (12). ■

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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1) Faculty of informatics, FON University, Skopje, Macedonia
E-mail address: anevskak@gmail.com
2) Faculty of informatics, FON University, Skopje, Macedonia
E-mail address: risto.malceski@gmail.com