Математички Билтен
Vol. 40(LXVI) No. 3
2016 (43-49)
Скопје, Македонија

ISSN 0351-336X (print)
ISSN 1857-9914 (online)
UDC: 515.124.3

ON (3,2,ρ)-*S*-*K*-METRIZABLE SPACES

Sonja Čalamani[1], Dončo Dimovski[2]
Marzanna Seweryn-Kuzmanovska[3]

Abstract. For a given $(3,2,\rho)$- metric d on a set M, we show that any $(3,2,\rho)$-*S*-*K*-metrizable space has an open refinement which is both locally finite and σ-discrete

1. INTRODUCTION

If we review historically the geometric properties, their axiomatic classification and the generalization of metric spaces we can see that, they have been subject of interest of great number of mathematicians and from their work a lot of have been developed. We will mention some of them: K. Menger ([14]), V. Nemytzki, P. S. Aleksandrov ([16], [1]), Z. Mamuzic ([13]), S. Gähler ([11]), A. V. Arhangelskii, M. Choban, S. Nedev ([2], [3], [17]), R. Kopperman ([12]), J. Usan ([18]), B. C. Dhage, Z. Mustafa, B. Sims ([6], [15]). The notion of (n,m,ρ)-metric is introduced in [7]. Connections between some of the topologies induced by a $(3,1,\rho)$-metric d and topologies induced by a pseudo-o-metric, o-metric and symmetric are given in [8]. For a given $(3,j,\rho)$-metric d on a set M, $j\in\{1,2\}$, seven topologies $\tau(G,d),\tau(H,d),\tau(D,d),\tau(N,d),\tau(W,d),\tau(S,d)$ and $\tau(K,d)$ on M, induced by d, are defined in [4], and several properties of these topologies are shown.

In this paper we consider only the topologies $\tau(S,d)$ and $\tau(K,d)$ induced by a $(3,2,\rho)$-metric d and for $\tau=\tau(S,d)=\tau(K,d)$ we prove that any open cover of a $(3,2,\rho)$-*S*-*K*-metrizable space (M,τ) has: a) an open refinement which is both locally finite and σ-discrete, b) σ-discrete base, and c) a $(3,2,\rho)$-*S*-K-metri-

2010 *Mathematics Subject Classification.* 54A10, 54E35, 54E99
Key words and phrases. $(3,2,\rho)$-metric, $(3,2,\rho)$-*S*-K-metrizable spaces

zable space (M,τ) is perfectly normal.

2. SOME PROPERTIES OF $(3,2,\rho)$ -*S*-*K*-METRIZABLE SPACES

In this part we state the notions (defined in [4]) used later.

Let M be a nonempty set, and let $d:M^3 \to \mathbb{R}_0^+ = [0,\infty)$. We state four conditions for such a map.

($\boldsymbol{M}0$) $d(x,x,x)=0$, for any $x \in M$;

($\boldsymbol{P}$) $d(x,y,z)=d(x,z,y)=d(y,x,z)$, for any $x,y,z \in M$;

($\boldsymbol{M}1$) $d(x,y,z) \le d(x,y,a)+d(x,a,z)+d(a,y,z)$, for any $x,y,z,a \in M$; and

($\boldsymbol{M}2$) $d(x,y,z) \le d(x,a,b)+d(a,y,b)+d(a,b,z)$, for any $x,y,z,a,b \in M$.

For a map d as above let $\rho = \{(x,y,z) \mid (x,y,z) \in M^3, d(x,y,z)=0\}$. The set ρ is a $(3,j)$-equivalence on M, as defined and discussed in [7], [4]. The set $\Delta = \{(x,x,x) \mid x \in M\}$ is a $(3,j)$-equivalence on M, $j=1,2$, and the set $\nabla = \{(x,x,y) \mid x,y \in M\}$ is a $(3,1)$-equivalence, but it is not a $(3,2)$-equivalence on M. The condition $(\boldsymbol{M}0)$ implies that $\Delta \subseteq \rho$.

Definition 1. Let $d:M^3 \to \mathbb{R}_0^+ = [0,\infty)$ and ρ be as above. If d satisfies $(\boldsymbol{M}0)$, $(\boldsymbol{P})$ and $(\boldsymbol{M}2)$, we say that d is a $(3,2,\rho)$-metric on M.

Let d be a $(3,2,\rho)$-metric on M, $x \in M$ and $\varepsilon > 0$. As in [4], we consider the following $\varepsilon-$ball, as subsets of M:

$L(x,\varepsilon) = \{y | y \in M, d(x,y,y) < \varepsilon\}$ -“little” ε-ball with center in x and radius ε.

Among the others, a $(3,2,\rho)$-metric d on M induces the following topologies as in [4]:

1) $\tau(K,d)$-the topology generated by all the ε-balls $L(x,\varepsilon)$, i.e. the topology whose base is the set of the finite intersections of ε-balls $L(x,\varepsilon)$;
2) $\tau(S,d)$-the topology defined by: $U \in \tau(S,d)$ iff $\forall x U$, $\exists \varepsilon > 0$ such that $L(x,\varepsilon) \subseteq U$.

Proposition 1. The ball $L(x,\varepsilon) \in \tau(S,d)$, for any x on M and $\varepsilon > 0$.

Proof. It is enough to show that for any $y \in L(x,\varepsilon)$ there is $\delta > 0,$ such that $L(y,\delta) \subseteq L(x,\varepsilon).$ Let $y \in L(x,\varepsilon)$ and $\delta = (\varepsilon - d(x,y,y))/4.$ Then, for any $z \in L(y,\delta)$ we have:

$$\begin{aligned} d(x,z,z) &\le d(x,y,y) + 2d(z,y,y) \\ &\le d(x,y,y) + 4d(y,z,z) \\ &< d(x,y,y) + 4\delta = \varepsilon. \end{aligned}$$

This implies that $z \in L(x,\varepsilon),$ i.e. $z \in L(y,\delta) \subseteq L(x,\varepsilon).$

From the proposition 1, it follows that $\tau(S,d) = \tau(K,d)$, for any $(3,2,\rho)$-metric d on M.

Definition 2. We say that a topological space (M,τ) is $(3,2,\rho)$-*S-K-metrizable* via a $(3,2,\rho)$-metric d on M, if $\tau = \tau(S,d) = \tau(K,d)$.

In the following theorem, one of the most important properties of $(3,2,\rho)$-S-K-metrizable space is established.

Let (M,τ) be a $(3,2,\rho)$-S-K-metrizable topological space.

Proposition 2. *Any open cover of a* $(3,2,\rho)$-*S-K-metrizable space has an open refinement which is both locally finite and* σ*-discrete.*

Proof. Let d be a $(3,2,\rho)$-metric on M and $\tau = \tau(S,d) = \tau(K,d).$ Let $\mathcal{U} = \{U_s \mid s \in S\}$ be an open cover of a $(3,2,\rho)$-S-K-metrizable space (M,τ), and let < be a well-ordering relation on the set S. Define inductively families $\mathcal{V}_n = \{V_{s,n} | s \in S, n \in \mathbb{N}\}$ of subsets of (M,τ) by letting

$$V_{s,n} = \cup L(x, 1/10^n),$$

where the union is taken over all points $x \in M$ satisfying the following conditions:

$$s \text{ is the smallest element of } S \text{ such that } x \in U_s, \tag{1}$$

$$x \notin V_{t,j} \text{ for } j < n \text{ and } t \in S, \tag{2}$$

$$L(x, 11/10^n) \subseteq U_s. \tag{3}$$

It follows from the definition of $V_{s,n}$ that the sets $V_{s,n}$ are open, and (3) implies that $V_{s,n} \subseteq U_s$. Let $y \in M$. Let s be the smallest element of S such that

$y \in U_s$. Then there is $n \in \mathbb{N}$ such that $L(y, 11/10^n) \subseteq U_s$. It is clear that, we have $y \in V_{t,j}$ for $j < n$ and a $t \in S$ or $y \in V_{s,n}$. Hence, the union $\mathcal{V} = \cup\{\mathcal{V}_n \mid n \in \mathbb{N}\}$ is an open refinement of the cover $\mathcal{U} = \{U_s \mid s \in S\}$.

We will prove that for any $n \in \mathbb{N}$ if $y_1 \in V_{s_1,n}$, $y_2 \in V_{s_2,n}$, and $s_1 \neq s_2$, then

$$d(y_1, y_2, y_2) > 1/10^n \text{ and } d(y_1, y_1, y_2) > 1/10^n, \tag{4}$$

and this will show that the families $\mathcal{V}_n$ are discrete, because any $1/10^{n+1}$-L-ball meets at most one member of $\mathcal{V}_n$.

Let $s_1 < s_2$. By the definition of $V_{s_1,n}$ and $V_{s_2,n}$ there are points x_1 and x_2 satisfying (1), (2) and (3) given above, such that $y_1 \in L(x_1, 1/10^n) \subseteq V_{s_1,n}$ and $y_2 \in L(x_2, 1/10^n) \subseteq V_{s_2,n}$. From (3) it follows that $L(x_1, 11/10^n) \subseteq U_{s_1}$, and from (1) we see that $x_2 \notin U_{s_1}$. Hence, $d(x_1, x_2, x_2) \geq 11/10^n$. The inequalities

$$\begin{aligned} 11/10^n &\leq d(x_1, x_2, x_2) \leq d(x_1, y_1, y_1) + 2d(x_2, y_1, y_1) \\ &\leq d(x_1, y_1, y_1) + 2d(x_2, y_2, y_2) + 4d(y_1, y_2, y_2) \\ &< 3/10^n + 4d(y_1, y_2, y_2), \end{aligned}$$

imply that $d(y_1, y_2, y_2) > 2/10^n > 1/10^n$.

Also the inequalities,

$$\begin{aligned} 11/10^n &\leq d(x_1, x_2, x_2) \leq d(x_1, y_1, y_1) + 2d(x_2, y_1, y_1) \\ &\leq d(x_1, y_1, y_1) + 2d(x_2, y_2, y_2) + 4d(y_1, y_2, y_2) \\ &< 3/10^n + 4d(y_1, y_2, y_2) \leq 3/10^n + 8d(y_1, y_1, y_2), \end{aligned}$$

imply that $d(y_1, y_1, y_2) > 1/10^n$.

From the latter follows the proof of (4).

Furthermore, it is enough to show that for each $t \in S$ and for each pair $k, j \in \mathbb{N}$,

$$\text{if } L(y, 1/10^k) \subseteq V_{t,j} \text{ then } L(y, 1/10^{k+j}) \cap V_{s,n} = \emptyset$$
$$\text{for } n \geq k + j \text{ and } s \in S. \tag{5}$$

From the definition of $V_{s,n}$ we have $V_{s,n} \subseteq V_{s,n+1}$, then $V_{t,j} \subseteq V_{t,j+1}$ for each $t \in S$ and each $j \in \mathbb{N}$. From (2) it follows that each x of $V_{s,n} = \cup L(x, 1/10^n)$, $x \notin V_{t,j}$ for $j < n$. Hence, for $n \geq k + j > k$ and $L(y, 1/10^k) \subseteq$

$V_{t,j}$ it follows that for each x of the union $\cup L(x,1/10^n)$, $d(y,x,x)>1/10^k$. We will show that $L(y,1/10^{k+j})\cap V_{s,n}\neq\varnothing$ for $n<k+j$. Let $y\in M$, then there are k,j,t such that $L(y,1/10^k)\subseteq V_{t,j}$ and $n<k+j$. For $\mathcal{V}_m$ discrete family and $m<k+j$, there is δ_m such that $L(y,\delta_m)\cap V_{s,m}\neq\varnothing$ for one s,m. Let $\delta=min\{\delta_m|1\le m<k+j\}$ then $L(y,\delta)\cap V_{s,m}\neq\varnothing$ for all $m<k+j$, i.e. $L(y,1/10^{k+j})\cap V_{s,n}\neq\varnothing$ for $n<k+j$. From the latter it follows that $\mathcal{V}_n$ is σ-discrete and locally finite. Hence, $\mathcal{V}=\cup\{\mathcal{V}_n|n\in\mathbb{N}\}$ is σ-discrete and locally finite.

Proposition 3. Any $(3,2,\rho)$-*S*-*K*-metrizable space (M,τ) has a σ-discrete base.
Proof. Let d be a $(3,2,\rho)$-metric on M and $\tau=\tau(S,d)=\tau(K,d)$. For $n\in\mathbb{N}$, let $\mathcal{U}_n=\{L(x,1/n)|xM\}$ be an open cover of M and let $\mathcal{V}_n$ be σ-discrete refinement obtained in proposition 2. The definition of $\tau(S,d)$ and $\tau(K,d)$ implies that $\mathcal{U}=\cup\{\mathcal{U}_n|n\in\mathbb{N}\}$ is a base for $\tau=\tau(S,d)=\tau(K,d)$. Since each $\mathcal{V}_n$ is σ-discrete refinement of $\mathcal{U}_n$, it follows that $\mathcal{V}=\cup\{\mathcal{V}_n|n\in\mathbb{N}\}$ is σ-discrete refinement of $\mathcal{U}$. Hence, $\mathcal{V}$ is σ-discrete base of (M,τ).

Corollary 1. Any $(3,2,\rho)$-*S*-*K*-metrizable space (M,τ) has a σ-locally finite base.

We will prove that the existence of a σ-locally finite base is also sufficient for metrizability of a $(3,2)$-*S*-*K*-*metrizable space* (M,τ).

Proposition 4. Any $(3,2)$-*S*-*K*-metrizable space (M,τ) is perfectly normal.
Proof. Let d be a $(3,2)$-metric on M and $\tau=\tau(S,d)=\tau(K,d)$. Let $\mathcal{V}=\cup\{\mathcal{V}_n|n\in\mathbb{N}\}$, where the families $\mathcal{V}_n$ are a locally finite, be a base for a space (M,τ). Consider an arbitrary open set $W\subseteq M$. For any $x\in W$ there is a natural number $n(x)$ and an open set $U_x\in\mathcal{V}_{n(x)}$ such that $x\in U_x\subseteq\overline{U_x}\subseteq W$. Letting $W_n=\cup\{U_x|n(x)=n\}$ we obtain a sequence $W_1,W_2,\cdots W_n$, $n\in\mathbb{N}$ of open subsets of M such that $W=\cup\{W_n|n\in\mathbb{N}\}$ and by property: if $\{A_s\}_{s\in S}$ is a locally finite family, then the family $\{\overline{A_s}\}_{s\in S}$ also is locally finite, we have

$\overline{W_n} \subseteq W$ for $n \in \mathbb{N}$. Normality of (M,τ) is proven in [9]. Since $W = \cup\{\overline{W_n} \mid n \in \mathbb{N}\}$, the space (M,τ) is perfectly normal.

Proposition 5. If $\rho = \Delta$, then (M,τ) is metrizable.

Proof. For $\rho = \Delta$, and the fact that (M,τ) is regular and has σ-locally finite base then from the metrization theorem of Nagata-Smirnov it follows that (M,τ) is metrizable.

References

[1] П. С. Александров, В.В.Немыцкий, *Условия метризуемости топологических простра- нств и аксиома симетрии,* Мат. сб. 3:3 (1938), 663-672

[2] А. В. Архангельский, *О поведении метризуемости при факторных тображениях,* ДАН 164, №о 2 (1965), 247-250

[3] М. Чобан, *О метрисуемыих пространствах,* Вестн.Моск. Ун-та, сер.Матем., мех, №о 3 (1959), 44-50

[4] S. Čalamani, D. Dimovski, *Topologies induced by (3,1,ρ)-metrics and (3,2,ρ)-metrics,* Inernational mathematical forum, Vol.9, no.21-24, (2014) 1075-1088

[5] S. Čalamani, T. Dimovski, D. Dimovski, *Separation properties for some topologies induced by (3,j,ρ)-metrics, j ∈{1,2}* FMNS, Vol.1, (2015) 24-30

[6] B.C. Dhage, *Generalized metric spaces and topological structure II,* Pure Appl.Math. Sci., 40 (1-2) (1994), 37-41

[7] D. Dimovski, *Generalized metrics - (n,m,r)-metrics,* Mat. Bilten, 16, Skopje (1992), 73-76

[8] D. Dimovski, *(3,1,ρ)-metrizable topological spaces,* Math. Macedonica, 3 (2005), 59-64

[9] T. Dimovski, D. Dimovski, *On (3,2,ρ)-K-metrizable spaces,* FMNS, (2013) 73-79

[10] R. Engelking, *General Topology,* (1977) Warsaw

[11] S. Gähler, *2-metrische räume und ihre topologische Struktur,* Math. Nachr. 26 (1963), 115-148

[12] R. Kopperman, *All topologies come from generalized metrics,* A. Math. Month. V 95 i2, 89-97

[13] Z. Mamuziċ, *Abstract distance and neighborhood spaces*, Proc. Prague Symp. (1962), 261-266

[14] K. Menger, *Untersuchungen über allgemeine Metrik,* Math. Ann. 100 (1928), 75-163

[15] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, Journal of Nonlinear and Convex Analysis, Vol. 7, Number 2 (2006), 289-297

[16] V. Nemytzki, *On the "third axiom of metric spaces",* Tr. Amer. Math. Soc. 29 (1927), 507-513

[17] С. Недев, *о-Метризуемые пространства*, Тр. Моск. Мат. Общ. Том 24 (1971), 201-236

[18] J. Usan, *<Nm,E>-seti s (n+1)-rastojaniem*, Review of Research, PMF, Novi Sad, Ser. Mat. 17 (1989), 2, 65-87

1) Faculty of technical sciences, University "St. Kliment Ohridski", Bitola, Republic of Macedonia
E-mail address: scalamani@yahoo.com

2) Faculty of Natural Sciences & Mathematics, Ss. Cyril and Methodius University, Skopje, Republic of Macedonia
E-mail address: donco@pmf.ukim.mk

3) Faculty of Education, University "St. Kliment Ohridski", Bitola, Republic of Macedonia
E-mail address: marzanna.kuzmanovska@yahoo.com