On ultraspherical polynomials

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Summary - *This paper will give several results involving the ultraspherical polynomials. The explicit expression for the product of these polynomials as a sum of such polynomials will be obtained. In particular Bailey's formula for the product of two associated Legendre functions is proved. Besides, some integrals involving ultraspherical polynomials have been evaluated.*

1. In hypergeometric notation, the ultraspherical polynomials may be defined by

$$(1)\quad P_n^{(\alpha,\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_2F_1\left(-n,\ n+2\alpha+1,\ \alpha+1,\ \frac{1-x}{2}\right),$$

with

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\ldots(a+n-1),$$

$$(a)_0 = 1.$$

Writing

$$(2)\quad {}_2F_1\left(-m,\ m+2\beta+1,\ \beta+1,\ \frac{1-x}{2}\right){}_2F_1\left(-n,\ n+2\alpha+1,\ \alpha+1,\ \frac{1-x}{2}\right)$$

$$= \sum_{k=0} a_k\, {}_2F_1\left(-m-n+2k,\ m+n+2\alpha-2k+1,\ \alpha+1,\ \frac{1-x}{2}\right),$$

we evaluate the coefficients a_k by a direct method.

We find

$$a_k = \frac{(n+\alpha-k+1)_m(n+2\alpha+1)_{m-k}(m+2\beta+1)_m(-n)_k(-m)_k(\beta+{}^1/_2)_k}{(2n+2\alpha-2k+1)_{2m}(2m+2\beta-2k+1)_{2k}(\beta+1)_{m-k}2^{-2k}k!}\ \cdot$$

$$\cdot\ \frac{2m+2n+2\alpha-4k+1}{2m+2n+2\alpha-2k+1}\, {}_4F_3\left[\begin{matrix}\beta-\alpha,\ k-m,\ {}^1/_2-\alpha,\ -k, \\ n-k+1,\ k-m-n-2\alpha,\ {}^1/_2+\beta\end{matrix}\ 1\right],$$

where

$$ {}_4F_3\left[\begin{matrix}\alpha, \beta, \gamma, \delta, \\ a, b, c\end{matrix}\; x\right] = \sum_{i=0}^{\infty} \frac{(\alpha)_i(\beta)_i(\gamma)_i(\delta)_i}{(a)_i(b)_i(c)_i} \frac{x^i}{i!}, $$

$$ m \leq n. $$

From (1) and (2) we have the formula

$$ \text{(3)} \quad P_n^{(\alpha,\alpha)}(x)P_m^{(\beta,\beta)}(x) = $$

$$ = (m+2\beta+1)_m \sum_{k=0}^{m} \binom{m+n-2k}{n-k} \cdot $$

$$ \cdot \frac{(n+\alpha-k+1)_k(m+\beta-k+1)_k(m+n+\alpha-2k+1)_k}{(2m+2\beta-2k+1)_{2k}(2n+2\alpha-2k+1)_{2m}2^{-2k}k!} \cdot $$

$$ \cdot (n+2\alpha+1)_{m-k}(\beta+{}^1/_2)_k \frac{2m+2n+2\alpha-4k+1}{2m+2n+2\alpha-2k+1} $$

$$ \cdot {}_4F_3\left[\begin{matrix}\beta-\alpha, k-m, {}^1/_2-\alpha, -k, \\ n-k+1, k-m-n-2\alpha, {}^1/_2+\beta\end{matrix}\; 1\right] P_{m+n-2k}^{(\alpha,\alpha)}(x), $$

which give the composition of the ultraspherical polynomials.

When $\alpha = \beta$, (3) becomes

$$ \text{(4)} \quad P_n^{(\alpha,\alpha)}(x)P_m^{(\alpha,\alpha)}(x) = $$

$$ = (m+2\alpha+1)_m \sum_{k=0}^{m} \binom{m+n-2k}{n-k} \cdot $$

$$ \cdot \frac{(n+\alpha-k+1)_k(m+\alpha-k+1)_k(m+n+\alpha-2k+1)_k}{(2m+2\alpha-2k+1)_{2k}(2n+2\alpha-2k+1)_{2m}2^{-2k}k!} \cdot $$

$$ \cdot (n+2\alpha+1)_{m-k}(\alpha+{}^1/_2)_k \frac{2m+2n+2\alpha-4k+1}{2m+2n+2\alpha-2k+1} P_{m+n-2k}^{(\alpha,\alpha)}(x). $$

If $\alpha = \beta = 0$, the formula (3) reduces to the well-known NEUMANN-ADAMS formula for LEGENDRE polynomials [1].

2. The connexion between the function $P_n^{(\alpha,\alpha)}(x)$ and the associated LEGENDRE function $P_n^r(x)$ is given by the equation

$$ \text{(5)} \quad P_{n-r}^{(r,r)}(x) = \frac{n!\,2^r}{(n+r)!}(1-x^2)^{-r/2}P_n^r(x), $$

(r is an integer).

Substituting $P_n^{(r,r)}(x)$ from (5) in (4) we get the remaining BAILEY's formula [2]

$$(1-x^2)^{-r/2} P_n^r(x) P_m^r(x) =$$

$$= 2^r \sum_{k=0}^{m-r} \frac{A_{m-k}^{-r} A_{k,-r} A_{n-k}^{-r}}{A_{m+n-r-k}^{r}} \frac{(m+n-2r-2k)!}{(m+n-2k)!} \cdot$$

$$\cdot \frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1} P^r_{m+n-r-2k}(x),$$

with

$$A_m^r = \frac{\left(\frac{1}{2}\right)_m}{(m+r)!}, \qquad A_{k,r} = \frac{\left(\frac{1}{2}-r\right)_k}{k},$$

$$m \leq n.$$

Furthermore, by means (3) one can deduce the formula for the product of two associated LEGENDRE functions of different degree and different order.

Really, we find

$$(1-x^2)^{-r/2} P_m^r(x) P_n^s(x) =$$

$$= 2^r \sum_{k=0}^{m-r} \frac{A_{m-k}^{-r} A_{k,-r} A_{n-k}^{-s}}{A_{m+n-r-k}^{s}} \frac{(m+n-r-s-2k)!}{(m+n-r+s-2k)!} \cdot$$

$$\cdot \frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1} {}_4F_3 \left[\begin{matrix} r-s,\ k+r-m,\ ^1/_2-s,\ -k, \\ n-k-s+1,\ k+r-m-n+s,\ ^1/_2+r, \end{matrix}\ 1\right] \cdot$$

$$P^s_{m+n-r-2k}(x),$$

$$(m-r \leq n-s).$$

3. Using (3) we can evaluate the integral

$$\int_{-1}^{1} P_n^{(\alpha,\alpha)}(x) P_m^{(\beta,\beta)}(x)\, dx.$$

Since

$$P_n^{(\alpha,\alpha)}(1) = (-1)^n P_n^{(\alpha,\alpha)}(-1) = \frac{(1+\alpha)_n}{n!},$$

and

$$\int_{-1}^{1} P_n^{(\alpha,\alpha)}(x)dx = \frac{1+(-1)^n}{n+2\alpha}\,\frac{2(\alpha)_{n+1}}{(n+1)!},$$

we have

$$\int_{-1}^{1} P_n^{(\alpha,\alpha)}(x) P_m^{(\beta,\beta)}(x)dx =$$

$$= 2K \sum_{k=0}^{m} \frac{(n+\alpha-k+1)_m (m+\beta-k+1)_k (n+2\alpha+1)_{m-k} (\beta+{}^1/_2)_k}{(2n+2\alpha-2k+1)_{2m} (2m+2\beta-2k+1)_{2k} (n-k)!\,(m-k)!\,2^{-2k} k!} \cdot$$

$$\cdot \frac{(m+n-2k+1)^{-1}}{(m+n+2\alpha-2k)} \cdot$$

$$\cdot \frac{2m+2n+2\alpha-4k+1}{2m+2n+2\alpha-2k+1}\, {}_4F_3 \left[\begin{matrix} \beta-\alpha,\ k-m,\ {}^1/_2-\alpha,\ -k, \\ n-k+1,\ k-m-n-2\alpha,\ {}^1/_2+\beta, \end{matrix}\ 1 \right],$$

$$K = [1+(-1)^{m+n}](\alpha)_{n+1}(m+2\beta+1)_m .$$

When $m = n$, and $\beta = 0$, we obtain

$$\int_{-1}^{1} P_n(x) P_n^{(\alpha,\alpha)}(x)dx = \frac{(n+2\alpha+1)_n}{2^{n-1}(2n+1)!!} .$$

If $\beta = 0$ and α is a positive integer, then

$$\int_{-1}^{1} P_m(x) P_n^{(\alpha,\alpha)}(x)dx = \binom{k+\alpha-1}{k} \frac{2^{\alpha}(m+n+2\alpha-1)!!}{(n+\alpha)_\alpha (m+n+1)!!},$$

$$n - m = 2k.$$

REFERENCES

[1]. Whittaker, E. T. and Watson G. N., *A Course of modern Analysis*, fourth edition, Cambridge, 1952.

[2]. Bailey W. N., *On the Product of two associated Legendre Functions*, « The Quarterly Journal of Mathematics », Oxford series, vol. II, N. 41, (1941), pp. 30-36.