ON SOME INTEGRALS INVOLVING LEGENDRE POLYNOMIALS

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In a previous paper (1), we have recently established the following result

$$(1)\quad \frac{d^r P_m(x)}{dx^r}\,\frac{d^s P_n(x)}{dx^s} = 2^r \sum_{k=0}^{m-r} \frac{A_{m-k}^{-r} A_{k,-r} A_{n-k}^{-s}}{A_{m+n-r-k}^{s}}\,\frac{(m+n-r-s-2k)!}{(m+n-r+s-2k)!}$$

$$\times \frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1}\,{}_4F_3\left[{r-s,\ k+r-m,\ \tfrac{1}{2}-s,\ -k, \atop n+1-s-k,\ k+r-m-n-s,\ \tfrac{1}{2}+r}\ 1\right]\frac{d^s P_{m+n-r-2k}(x)}{dx^s}$$

where $P_n(x)$ is Legendre polynomial and

$$A_s^n = \frac{\left(\frac{1}{2}\right)_s}{(n+s)!},\quad A_{k,r} = \frac{\left(\frac{1}{2}-r\right)_k}{k!},$$

$${}_4F_3\left[{\alpha,\ \beta,\ \gamma,\ \delta, \atop a,\ b,\ c,}\ x\right] = \sum_{i=0}^{\infty} \frac{(\alpha)_i(\beta)_i(\gamma)_i(\delta)_i}{(a)_i(b)_i(c)_i}\,\frac{x^i}{i!},$$

$$(\alpha)_i = \alpha(\alpha+1)\ldots(\alpha+i-1),\ (\alpha)_0 = 1,$$

$$(m-r \leqslant n-s).$$

Here we profit by this result to evalute certain definite integrals involving products of the derivatives of Legendre polynomials. Besides, some generalisations of this result have also been given.

It may be easily shown that

$$\int_{-1}^{1} \frac{d^r P_n(x)}{dx^r}\,dx = [1+(-1)^{n-r}]\,\frac{(n+r-1)!\,2^{1-r}}{(r-1)!\,(n-r+1)!}.$$

Then we have

$$(2)\quad I_{m,n;r,s}^{(-1,1)} = \int_{-1}^{1} \frac{d^r P_m(x)}{dx^r}\,\frac{d^s P_n(x)}{dx^s}\,dx$$

$$= K \sum_{k=0}^{m-r} \frac{A_{k,-r}\,A_{m-k}^{-r}\,A_{n-k}^{-s}}{A_{m+n-r-k}^{s}}\,\frac{(m+n-r+s-2k)^{-1}}{(m+n-r-s-2k+1)}$$

$$\times \frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1}\,{}_4F_3\left[{r-s,\ k+r-m,\ \tfrac{1}{2}-s,\ -k, \atop n+1-s-k,\ k+r-m-n-s,\ \tfrac{1}{2}+r,}\ 1\right],$$

$$K = [1 + (-1)^{m+n-r-s}]\,\frac{2^{r-s+1}}{(s-1)!},\quad (n - s \geqslant m - r).$$

For the special case of m, n, r and s, (2) can be expressed in a simpler form. For example, if $r = s$, then

$$I^{(-1,\,1)}_{m,\,n\,;\,r,\,r} = \frac{2^2}{(r-1)!}\sum_{k=0}^{m-r}\frac{A^{-r}_{m-r}\,A_{k,\,-r}\,A^{-r}_{n-r}}{A^{r}_{m+n-r-k}}\,\frac{(m+n-2k)^{-1}}{(m+n-2r-2k+1)}$$

$$\times\,\frac{2m+2n-2r-4k+1}{2m+2n-2r-2k+1},$$ for $m+n$ even, $I^{(-1,\,1)}_{m,\,n\,;\,r,\,r} = 0$, for $m+n$ odd.

From the above it is also easy to give

$$I^{(-1,\,1)}_{m,\,n\,;\,r,\,0} = \begin{cases} 2\binom{k+r-1}{r-1}\dfrac{(n+m+r-1)!!}{(n+m-r+1)!!}, & m-r-n = 2k \geqslant 0,\\[2mm] 0, & m-r-n < 0 \text{ or } m-n-r = 2k+1. \end{cases}$$

$$(1 \leqslant r \leqslant m).$$

Proceeding in a similar way, we get

$$(3')\quad I^{(0,\,1)}_{m,\,n\,;\,r,\,s} = \int_0^1 \frac{d^rP_m(x)}{dx^r}\,\frac{d^sP_n(x)}{dx^s}\,dx =$$

$$= \frac{2^{r-s+1}}{(s-1)!}\sum_{k=0}^{m-r} B^{r,\,s}_{m,\,n,\,k}\,\frac{(2\lambda+2s-2k-1)!}{(2\lambda-2k+1)!},$$

$$m+n-r-s = 2\lambda,\ m-r \leqslant n-s.$$

and

$$(3'')\quad I^{(0,1)}_{m,\,n\,;\,r,\,s} = 2^{r-s+1}\sum_{k=0}^{m-r} B^{r,\,s}_{m,\,n,\,k}\,(2\lambda+2s-2k-2)!$$

$$\times\left[\frac{1}{(s-1)!(2\lambda-2k)!} - \frac{(-1)^{\lambda-k}}{2^{2\lambda-2k}(2\lambda-2k+s-1)!}\binom{2\lambda-2k+s-1}{\lambda-k}\right]$$

$$m+n-r-s = 2\lambda+1,\ m-r \leqslant n-s,$$

where

$$B^{r,\,s}_{m,\,n,\,i} = \frac{A^{-r}_{m-i}\,A^{-s}_{n-i}\,A_{i,\,-r}}{A^{s}_{m+n-r-i}}\,\frac{(m+n-r-s-2i)!}{(m+n-r+s-2i)!}$$

$$\frac{2m+2n-2r-4i+1}{2m+2n-2r-2i+1}\;{}_4F_3\left[\begin{matrix} r-s,\ i+r-m,\ \tfrac12-s,\ -i, \\ n+1-i-s,\ i+r-m-n-s,\ \tfrac12+r \end{matrix}\;1\right]$$

Putting $s = 0$ in (3) we readily see that

$$\int_0^1 P_n(x)\,\frac{d^rP_m(x)}{dx^r}\,dx = 0,\ m+n-r = 2\lambda$$

$$= 2^r \sum_{k=0}^{m-r} (-1)^{\lambda-k} \frac{A_{n-k} A_{m-k}^{-r} A_{k,-r}}{A_{2\lambda-k+1}} \frac{(2\lambda-2k-1)!!}{(2\lambda-2k+2)!!}$$

$$\times \frac{4\lambda - 4k + 3}{4\lambda - 2k + 3} {}_4F_3 \left[{}^{r,\ k+r-m,\ \frac{1}{2},\ -k,}_{n-k+1,\ k+r-m-n,\ \frac{1}{2}+r} 1\right],$$

$$(m + n - r = 2\lambda + 1),\ m - r \leqslant n.$$

The result (1) can be used to obtain the product

$$\frac{d^p P_l(x)}{dx^p} \frac{d^r P_m(x)}{dx^r} \frac{d^s P_n(x)}{dx^s},$$

as a sum of the functions the same type, which formula is useful in some problems of quantum mechanics.

We obtain the formula

$$(4) \quad \frac{d^p P_l(x)}{dx^p} \frac{d^r P_m(x)}{dx^r} \frac{d^s P_n(x)}{dx^s}$$

$$= 2^{r+s} \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} B_{m,\,n,\,i}^{r,\,s}\, B_{m+n-r-2i,\,l,\,j}^{s,\,p} \frac{d^p P_{l+m+n-r-s-2i-2j}(x)}{dx^p},$$

$$\alpha = \min(m - r,\ n - s),$$

$$\beta = \min(m + n - r - s - 2i,\ l - p).$$

Starting with (4) we can find some other integral formulae for the product of tree derivatives of Legendre polynomials.

We obtain

$$I_{l,\,m,\,n\,;\,p,\,r,\,s}^{(-1,\,1)} =$$

$$K \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} B_{m,\,n,\,i}^{r,\,s}\, B_{m+n-r-2i,\,l,\,j}^{s,\,p} \frac{(l+m+n+p-r-s-2i-2j-1)!}{(l+m+n-p-r-s-2i-2j+1)!},$$

$$K = [1 + (-1)^{l+m+n-p-r-s}] \frac{2^{1-p+r+s}}{(p-1)!}.$$

If we put $p = r = s = 0$, we have the known formula ([2])

$$\int_{-1}^{1} P_l(x)\, P_m(x)\, P_n(x) dx = \frac{A_{k-l}\, A_{k-m}\, A_{k-n}}{A_k} \frac{2}{2k+1},$$

$$l + m + n = 2k.$$

RÉFÉRENCES

([1]) B. S. Popov, *Ann. de la Fac. de phil. Skopje, Section des sciences naturelles*, t. **10**, 1957, p. 1-20.

([2]) M. E. Feldheim, *Bulletin de la Société math. de France*, t. **LXVIII**, 1940, p. 204.