Математички Билтен
16 (XLII)
1992 (23-30)
Скопје, Македонија

MATHEMATICAL APPLICATIONS OF THE INDUCTION METHOD IN THE THEORY OF ABSTRACT STATIONARY EQUATIONS

I.V. Kulikov (USSR, Rostov-on-Don)

Abstract. We study abstract systems of stationary inequalities of the following type:

$$\sum_{1\le j\le m} a_{ij}\,\tilde{K}_{\tau_{ij}}(x_j) \le N_i,\ 1 \le i \le t, \qquad (1)$$

where m,t,N_i $(1 \le i \le t)$, a_{ij} $(1 \le i \le t,\ 1 \le j \le m)$ are the elements of the set Z_+ - the set of natural numbers.

In this article we formulate and prove theorems about the solutions of an abstract stationary equation $\tilde{K}_\eta(x)=r$, $r\in Z_+$, and observe [7] estimates for the number of solutions of (1).

Let Λ^o be the set of all functions $\eta(x) \equiv \eta$, $\eta\colon Z_+ \to Z_+$, which satisfies the condition $\eta(x) \le x$, when $x\in Z_+$. For every function $\eta\in\Lambda^o$ define its index $\tilde{K}_\eta(x) \equiv \tilde{K}_\eta$, $\tilde{K}_\eta\colon Z_+ \to Z_+$ by the equalities

$$\tilde{K}_\eta(x) := \min\{\ell : \ell\in Z_+,\ \eta^{(\ell)}(x)\in A_\eta\},\ x\in Z_+;$$

$$\eta^{(1)}(x) := \eta(x),\ \eta^{(\ell)}(x) := \eta(\eta^{(\ell-1)}(x)),\quad \ell \ge 2;$$

$$A_\eta := \{x : x\in Z_+,\ \eta(x) = x\}.$$

In the system (1) the functions $\tau_{ij}\in\Lambda^o$ $(1 \le i \le t,\ 1 \le j \le m)$, $\tilde{K}_{\tau_{ij}}$ is their index and

$$\forall i\ (i=\overline{1,t})\quad \sum_{1\le j\le m} a_{ij} \le N_i.$$

It follows therefore that the system (1) has a solution $x=(1,1,\dots,1)$, since $\tilde{K}_{\tau_{ij}}(1) = 1$.

We use the following definitions. P - the set of prime numbers, $d(x)$ - the number of positive divisors of $x\in Z_+$, $\psi_n(\eta)$ - the set of solutions of the stationary equation $\tilde{K}_\eta(x)=n$, $n\in Z_+$.

The main theorems

1°. Theorem 1. Let $\eta\in\Lambda^o$ and let the following conditions be true:

1. The function $\eta(p) \to \infty$, when $p\in P$ and $p \to \infty$.

2. For any m ($m \in Z_+$), $\eta(\eta(2n+1)) \neq \eta(2m+1)$.

3. The set A_η^{-1},

$$A_\eta^{-1} \overset{def}{=\!=} \{x: x \in Z_+, \eta(x) \in A_\eta\}, \tag{2}$$

is limited.

4. The following inequalities are satisfied

$$\tilde{K}_\eta(x)+\tilde{K}_\eta(y) \le \begin{cases} \tilde{K}_\eta(xy), & \text{if } (-1)^x+(-1)^y = 2; \\ \tilde{K}_\eta(xy)+1, & \text{if } (-1)^x+(-1)^y < 2. \end{cases} \tag{3}$$

Then for any n ($n \in Z_+$), $\psi_n(\eta) \neq \emptyset$ and is limited.

Theorem 2. Let $\eta \in \Lambda^o$, $C_1 \ge 2$, and let the following conditions be true:

1. The function $\eta(p) \to \infty$, when $p \in P$ and $p \to \infty$.

2. For any p ($p \in P$), $d(\eta(p)) \le C_1$.

3. The set A_η^{-1} is limited.

4. The inequalities (3) are satisfied.

Then for any n ($n \in Z_+$) the set

$$T_{n,C_1}(\eta) \overset{def}{=\!=} \{x: x \in Z_+, \tilde{K}_\eta(x) = n, d(x) \le C_1\} \tag{4}$$

is empty or $T_{n,C_1}(\eta) \neq \emptyset$ and is limited.

Another sufficient conditions will be given in the next Theorem 3. Therefore we shall define a subset Λ^* of Λ^o. For every function $\eta \in \Lambda^*$ the set $A_\eta = \{1\}$.

Let Λ^* be the set of functions $\eta(x) \equiv \eta$, $\eta: Z_+ \to Z_+$, which satisfies the conditions:

1. $\eta(1) = \eta(2) = 1$.

2. $\forall x$ ($x \in Z_+$, $x \ge 3$), $2 \le \eta(x) < x$.

The index $\tilde{K}_\eta(x)$ in Λ^* we denote by $K_\eta^*(x)$.

Theorem 3. Let $\eta \in \Lambda^*$ and let the following conditions be true:

1. There exists such C_2 ($0 < C_2 < 1$) that for any x ($x \in Z_+$)

$$C_2 d(x) \le d(\eta(x)) \le d(x) \tag{5}$$

2. For any m ($m \in Z_+$) there exists such C_m (≥ 2) that for any x ($x \in Z_+$), $d(x) \leq m$)

$$x \leq C_m \eta(x). \tag{6}$$

Then for any n ($n \in Z_+$), $\psi_n(n) = \emptyset$ or $\psi_n(\eta) \neq \emptyset$, is limited and for any pair y,m ($y \in Z_+, m \in Z_+, y \in \psi_n(\tau)$, $d(y) \leq m$) the following estimate is true

$$y \leq C_m^n. \tag{7}$$

Corollary. Let $\tau \in \Lambda^*$ and let be given the system of numbers $\{C_{ij}\}_{i,j=1}$, $C_{ij} \geq 2$, and the system of sets $\{A_{ij}\}_{i,j=1}$, $A_{ij} \neq \emptyset$, which satisfies the formulas:

$$\begin{aligned} &\forall x,i,j \ (x \in A_{ij},\ i \in Z_+,\ j \in Z_+) \\ &\tau(x) \in A_{ij}, \\ &x \leq C_{ij}\tau(x) \end{aligned} \tag{8}$$

Then for every pair i,j we have:

$$\forall n \ (n \in Z_+) \quad \psi_n(\tau) \cap A_{ij} = \emptyset$$

or $\psi_n(\tau) \cap A_{ij} \neq \emptyset$ and

$$\max\{x:\ x \in \psi_n(\tau) \cap A_{ij}\} \leq C_{ij}^n. \tag{9}$$

Proofs

2°. Proof of the Theorem 1. From the definition of $\tilde{K}_\eta(x)$, formula (3) and by the mathematical induction we have

$$\tilde{K}_\eta(2) = 1, \quad \forall u \ (\in Z_+) \quad \tilde{K}_\eta(2^u) \geq u.$$

Hence for any n ($\in Z_+$)

$$\psi_n(\eta) \overset{\text{def}}{=\!=} \{t:\ t \in Z_+,\ \tilde{K}_\eta(t) = n\} \neq \emptyset.$$

By formula $\psi_1(\eta) = \eta^{-1}(A_\eta)$ and condition 3, $\psi_1(\eta)$ is limited. Suppose that ($n \geq 1$)

$$\psi_1(\eta), \ldots, \psi_n(\eta)$$

are limited and investigate $\psi_{n+1}(\eta)$. Let

$$X_n \overset{\text{def}}{=\!=} \sum_{1 \leq j \leq n} \max\{t:\ t \in \psi_j(\eta)\}. \tag{10}$$

There is a number C_3 (≥ 1) such that

$$\forall p\ (p \in P,\ p > C_3)\ \eta(p) > X_n$$

(condition 1). Therefore, if $p > C_3$, then $\tilde{K}_\eta(p) \geq n+2$.

Let $z \in \psi_{n+1}(\eta)$

1. If $z \in P$, then

$$z \leq C_3. \tag{11}$$

2. If $z \notin P$, $z \equiv 1(2)$, then $z=xy$, $x \equiv 1 \equiv y(2)$, $x \geq 3$, $y \geq 3$. By condition 2 $\tilde{K}_\eta(x) \geq 2$, $\tilde{K}_\eta(y) \geq 2$. Then inequality (3) gives $2 \leq \tilde{K}_\eta(x) \leq n$, $2 \leq \tilde{K}_\eta(y) \leq n$. Using the assumption of induction we derive

$$x < X_n,\ y < X_n,\ z = xy < X_n^2. \tag{12}$$

3. If $z \notin P$, $z \equiv 0(2)$, then $z=xy$, $x \geq 2$, $y \geq 2$.

α) Let $x \equiv 0(2)$, $y \equiv 1(2)$. By (3) $\tilde{K}_\eta(y) \leq n+1$. Using condition 2 $\tilde{K}_\eta(y) \geq 2$ and $\tilde{K}_\eta(x) \leq n$. Therefore

$$x \leq X_n,\ y < C_3 + X_n^2,$$
$$z = xy < X_n(C_3+X_n^2). \tag{13}$$

β) Let $x \equiv y \equiv 0(2)$. By inequality (3) we get $\tilde{K}_\eta(x) \leq n$, $\tilde{K}_\eta(y) \leq n$. Using the assumption of induction $x \leq X_n$, $y \leq X_n$ and

$$z \leq X_n^2 \tag{14}$$

The inequalities (11), (12), (13), (14) proves

$$z < C_3X_n + X_n^3$$

and we obtain Theorem 1.

Proof of the Theorem 2. By condition 3

$$\emptyset \neq T_{1,C_1}(\eta) \subset \psi_1(\eta) = \eta^{-1}(A_\eta)$$

and $T_{1,C_1}(\eta)$ is limited. Suppose that for any j ($j=\overline{1,n}; n \geq 1$) $T_{j,C_1}(\eta) = \emptyset$ or $T_{j,C_1}(\eta) \neq \emptyset$ and is limited. Investigate $T_{n+1,C_1}(\eta)$.

Let

$$X_n' \overset{def}{=\!=} \max\{t:\ t \in \bigcup_{1 \leq j \leq n} T_{j,C_1}(\eta)\}. \tag{15}$$

There is a number C_4 (≥ 1) such that

$$\forall p \ (p \in P, p > C_4) \quad \eta(p) > X_n'$$

(condition 1). Therefore, from the condition 2

$$\eta(p) \in \bigcup_{j=n+1}^{\infty} T_{j,C_1}(\eta)$$

and $\tilde{K}_\eta(p) \geq n+2$.

Let $T_{n+1,C_1}(\eta) \neq \emptyset$ and $z \in T_{n+1,C_1}(\eta)$.

1. If $z \in P$, then

$$z \leq C_4. \tag{16}$$

2. Let $z \notin P$ and $z \equiv 1(2)$. By p we denote some prime divisor of z, $z=ps$, $s \in Z_+$. From (3) $\tilde{K}_\eta(p) \leq \tilde{K}_\eta(ps) = n+1$. Using (15), (16) and the assumption of mathematical induction, we obtain

$$p < X_n' + C_4.$$

Since $d(z) \leq C_1$, then

$$z < (X_n'+C_4)^{d(z)} \leq (X_n'+C_4)^{C_1}. \tag{17}$$

3. Let $z \in P$ and $z \equiv 0(2)$.

α) If $z = 2^\ell y$, where $\ell \geq 1$, $y \geq 3$, $y \equiv 1(2)$, then by (3)

$$\tilde{K}_\eta(2^\ell) + \tilde{K}_\eta(y) \leq \tilde{K}_\eta(z)+1 = n+2.$$

But $\tilde{K}_\eta(2^\ell) \geq 1$ and, therefore, $\tilde{K}_\eta(y) \leq n+1$. Besides this $d(y) < d(z) \leq C_1$. If $1 \leq \tilde{K}_\eta(y) \leq n$, then by induction $y \leq X_n'$. If $\tilde{K}_\eta(y) = n+1$, then for prime y (item 1) and multiple y (item 2) we have the following considerable estimates

$$y \leq C_4, \ y < (X_n'+C_4)^{C_1}.$$

Hence

$$y < (X_n'+C_4)^{C_1}.$$

Since $\ell < C_1$, we have

$$z = 2^\ell y < 2^{C_1}(X_n'+C_4)^{C_1}. \tag{18}$$

β) If $z = 2^\tau$, $\tau \geq 2$, then from the inequality $\tau < d(z) \leq C_1$ it follows

$$z < 2^{C_1} \tag{19}$$

The inequalities (16), (17), (18), (19) prove

$$z < 2^{C_1}(X_n'+C_4)^{C_1}$$

and the Theorem 2.

Proof of the Theorem 3. The set $\psi_1(\eta)=\{1,2\}$. Therefore theorem is true for n=1. Suppose that the theorem is true for some n ($\in Z_+$) and consider $n+1 \geq 2$. Let $\psi_{n+1}(\eta) \neq \emptyset$ and $y \in \psi_{n+1}(\eta)$. Then $\eta(y) \in \psi_n(\eta) \neq \emptyset$. By assumption of mathematical induction $\psi_n(\eta)$ is limited. Then there exists $b_n = b_n(\eta)$ ($\in Z_+$) such that

$$\forall t\ (\in \psi_n(\eta))$$
$$t \leq b_n(\eta),\ d(t) \leq b_n(\eta). \tag{20}$$

By (5) we have

$$d(y) \leq C_2^{-1} d(\eta(y)) \leq C_2^{-1} b_n(\eta),$$
$$d(y) \leq \left[C_2^{-1} b_n(\eta)\right].$$

It is proved, that the function d(y) is uniformly bounded on $\psi_{n+1}(\eta)$. It follows from (6) and (20) that

$$y \leq C_{[C_2^{-1}b_n(\eta)]}^{\ \eta(y)} \leq C_{[C_2^{-1}b_n(\eta)]}^{\ b_n(\eta)}$$

Consequently $\psi_{n+1}(\eta)$ is limited.

We shall prove (7). Let $y \in \psi_{n+1}(\eta)$ and $d(y) \leq m \in Z_+$. Then by (5)

$$d(\eta(y)) \leq d(y) \leq m$$

and $\eta(y) \in \psi_n(\eta)$. Using (6) and assumption of induction for $\eta(y)$, we obtain

$$y \leq C_m \eta(y) \leq C_m \cdot C_m^n = C_m^{n+1}$$

and the Theorem 3.

Corollary may be proved by analogous method.

Applications

3^o. We shall illustrate some mathematical applications of results in the theory of stationary inequalities.

Let Λ_1 be the set of all functions $\eta \in \Lambda^*$, which satisfies the condition

$$K^*_\eta(x)+K^*_\eta(y) \le \begin{cases} K^*_\eta(xy), & \text{if } (-1)^x+(-1)^y = 2; \\ K^*_\eta(xy)+1, & \text{if } (-1)^x+(-1)^y < 2. \end{cases}$$

Suppose that the $m,t,a_{i,j}$ $(1 \le i \le t,\ 1 \le j \le m)$, N_i $(1 \le i \le t)$ are natural numbers and for any i $(1 \le i \le t)$, $\sum_{1\le j\le m} a_{ij} \le N_i$. Let $\tau_{ij} \equiv \tau_{ij}(x)$, $(1 \le i \le t,\ 1 \le j \le m)$ are elements of Λ_1.

We consider the following system of stationary inequalities

$$\begin{array}{l} a_{11}K^*_{\tau_{11}}(x_1)+a_{12}K^*_{\tau_{12}}(x_2)+\ldots+a_{1m}K^*_{\tau_{1m}}(x_m) \le N_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{t1}K^*_{\tau_{t1}}(x_1)+a_{t2}K^*_{\tau_{t2}}(x_2)+\ldots+a_{tm}K^*_{\tau_{tm}}(x_m) \le N_t. \end{array} \qquad (21)$$

The vector $(x_1,x_2,\ldots,x_m)\in Z^m_+$, which satisfies (21) is called the solution of (21).

In [7], according to properties of the function $K^*_\eta(x)$, $\eta\in\Lambda_1$, the following Theorem was proved.

Theorem 4. Let for any i,j $(1 \le i \le t,\ 1 \le j \le m)$ there exists C_{ij} $(C_{ij} \ge 2)$ such that for any prime p

$$p \le C_{ij}\tau_{ij}(p).$$

Then the set of solutions of the system (21) is limited. The number A of all solutions of the system (21) satisfies the inequality

$$A \le \min_{1\le i\le t} \prod_{1\le j\le m} 2^{2[a^{-1}_{ij}N_i]-1} \cdot (C_{ij}^{[a^{-1}_{ij}N_i]-1}),$$

where $[a^{-1}_{ij}N_i]$ is the entire part of $a^{-1}_{ij}N_i$.

Basing on the Theorem 1-3 we can consider the systems (1) and receive estimates for the number of solutions of the systems (1). Also may be received the bounds of these solutions, which with the help of electronic machins solves the problem of finding all solutions of systems (1) for the small numbers m,t,a_{ij},N_i.

REFERENCES

[1] Shapiro, H.: An arithmetic function arising from the ϕ function, Amer. Math. Monthly, 1943, 50: 18-30

[2] Mills, W.H.: Iteration of the ϕ function, Amer. Math. Monthly, 1943, 50: 5, 547-549

[3] Parnami, J.C.: On iterates of Euler's ϕ function, Amer. Math. Monthly, 1970, 77: 1, 60-61

[4] Куликов, И.В.: Об общем стационарном уравнении целозначной функции, Ростов ун-т., Ростов н/Д, 1989, 11 с. Библиогр. 10 назв., Рус. Деп. в. ВИНИТИ 26.06.89, N° 4221-В89

[5] Куликов, И.В.: К теории суперпозиций мультипликативной функции. II. О стационарном уравнении индекса, Ростов ун-т., Ростов н/Д, 1988, 12 с., Библиогр. 5 назв., Рус. Деп. в ВИНИТИ 25.11.88, N° 8342-В88

[6] Куликов, И.В.: К теории суперпозиций мультипликативной функции. I. Мультипликативные функции в евклидовом кольце, Краснодар. политехн. ин-т., Краснодар, 1986, 16 с. Библиогр. 7 назв., Рус. Деп. в ВИНИТИ 16.07.86, N° 5181-В86

[7] Куликов, И.В.: О совмесности систем стационарных неравенств. Известия Северо-Кавказкого научного центра высшей школы, Естеств. науки, 1990, N° 4

ПРИМЕНИ НА МЕТОДОТ НА ИНДУКЦИЈА ВО ТЕОРИЈАТА НА АПСТРАКТНИ СТАЦИОНАРНИ РАВЕНКИ

И.В. Куликов

Резиме

Се разгледуваат одредени апстрактни системи стационарни неравенки, се формулираат и докажуваат теореми за решенијата на соодветни апстрактни стационарни равенки и се прават оценки за бројот на решенија на неравенките.