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COHERENT CATEGORY Coh

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A new coherent category of coherent inverse systems will be defined which from the earlier known category COH defined in [6] and [7] differs in the definition of coherent homotopy.

The obtained category of coherent inverse systems $\underline{X}=(X_a,p_{a_0\ldots a_n},A)$ with this new definition of coherent homotopy satisfies the requirements of the theory announced in [1] and [2] without explicitely given formulas.

For inverse systems $(X_a,p_{a_0a_1},A)$, which are special case of coherent inverse systems such a theory is developed in [4] (also [3]).

First we present the notion of a coherent inverse system.

An ordered set $(A,<)$ is directed if for every $a,a'\in A$ there exists $a''\in A$ such that $a''>a, a''>a'$. Further on, $(A,<)$ will be a cofinite directed set i.e. each element of A has only a finite number of predecessors.

A coherent inverse system $\underline{X}=(X_a,p_{\underline{a}},A)$ consists of the following: a directed set $(A,<)$, for every $a\in A$ a topological space X_a, for $a_0 < a_1$ a map $p_{a_0a_1}: X_{a_1} \to X_{a_0}$ and for $n > 1$ and $\underline{a}=(a_0,\ldots,a_n)$, $a_0 < \ldots < a_n$ a sequence in A, a map $p_{\underline{a}}:I^{n-1}\times X_{a_n}\to X_{a_0}$ such that

$$p_{\underline{a}}(t,x)=\begin{cases} p_{a_0\ldots\hat{a}_i\ldots a_n}(t_1,\ldots,\hat{t}_i,\ldots,t_{n-1},x), & t_i=0 \\ p_{a_0\ldots a_i}(t_1,\ldots,t_{i-1},p_{a_i\ldots a_n}(t_{i+1},\ldots,t_{n-1},x)), & t_i=1 \end{cases} \quad (1)$$

where $t=(t_1,\ldots,t_{n-1})\in I^{n-1}$, $x\in X_{a_n}$, and $1\le i\le n-1$. As usually $\hat{a}_i$ means that a_i is omitted i.e. $(a_0,\ldots,\hat{a}_i,\ldots,a_n)=$

$=(a_0,\ldots,a_{i-1},a_{i+1},\ldots,a_n)$. For n=2, $p_{a_0a_1a_2}: I\times X_{a_2} \to X_{a_0}$ is a homotopy connecting maps $p_{a_0a_2}$ and $p_{a_0a_1}p_{a_1a_2}$ and for n=3 the map $p_{a_0\ldots a_3}: I^2\times X_{a_3} \to X_{a_0}$ satisfies the boundary conditions showed in the figure 1 (where for simplicity of the notation $p_{0\ldots n}=p_{a_0\ldots a_n}$).

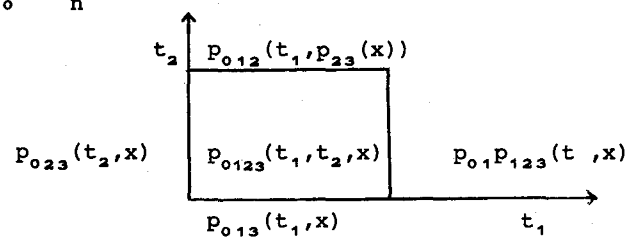

Fig. 1

For a given integer $n > 0$ and a sequence of integers $\underline{j}=(j_0,\ldots,j_k)$, $0=j_0 < \ldots < j_k=n$, we define a subset $I^{\underline{j}}$ of the n-dimensional cube $I^n=[0,1]^n$ by

$$I^{\underline{j}} = \{t=(t_1,\ldots,t_n):\ 1 \geq t_{j_1} \geq t_{j_2} \geq \ldots \geq t_{j_k} \geq 0\} \qquad (2)$$

By this definition $I^{01}=I$, and $I^{on}=I^n$.

If B is a directed set and $\phi: B \to A$ a strictly increasing function we put $\phi(b_0,\ldots,b_n)=(\phi(b_0),\ldots,\phi(b_n))$ for any sequence $(b_0,\ldots,b_n)$, $b_0 < \ldots < b_n$ in B.

A coherent map $f: \underline{X} \to \underline{Y}=(Y_b,q_b,B)$ consists of the following

i) A strictly increasing function $\phi: B \to A$

ii) For n=0, and $b_0 \in B$ of a map $f_{b_0}: X_{\phi(b_0)} \to Y_{b_0}$.

For $n > 1$, and $\underline{b}=(b_0,\ldots,b_n)$, $b_0 < \ldots < b_n$ a sequence in B, and $\underline{j}=(j_0,\ldots,j_k)$, $0=j_0 < \ldots < j_k=n$, a sequence of integers, of a map $f_{\underline{b}}^{\underline{j}}: I^{\underline{j}}\times X_{\phi(b_n)} \to Y_{b_0}$ satisfying the following boundary conditions

$$f_{\underline{b}}^{\underline{j}}(t_1,\dots,t_n,x) = \qquad (2)$$

$$\begin{cases} q_{b_0\dots b_{j_1}}(t_1,\dots,t_{j_1-1},f_{b_{j_1}\dots b_n}^{j_1-j_1\dots j_k-j_1}(t_{j_1+1},\dots,t_n,x)), & t_{j_1}=1 \\ f_{\underline{b}}^{j_0\dots\hat{j}_i\dots j_k}(t_1,\dots,t_{j_i-1},1,t_{j_i+1},\dots,t_n,x), & t_{j_i}=t_{j_i+1},\ 0<i<k \\ f_{b_0\dots b_{j_{k-1}}}^{j_0\dots j_{k-1}}(t_1,\dots,t_{j_{k-1}-1},p_\phi(b_{j_{k-1}}\dots b_n)(t_{j_{k-1}+1},\dots,t_{n-1},x)), & t_n=0 \\ f_{b_0\dots\hat{b}_j\dots b_n}^{j_0\dots j_i j_{i+1}-1\dots j_i-1}(t_1,\dots,\hat{t}_j,\dots,t_n,x), & t_j=0,\ i_i<j<j_{i+1} \end{cases}$$

Specially, for n=1 and $b_0 < b_1$ the map (homotopy $f_{b_0b_1}:I\times X_{\phi(b_1)}\to X_{b_0}$ satisfies $f_{b_0b_1}(1,x)=q_{b_0b_1}f_{b_1}(x)$, $f_{b_0b_1}(0,x)=f_{b_0}p_{\phi(b_0b_1)}(x)$.
For n=2 the boundary conditions of the maps $f_{b_0b_1b_2}:I^{02}\times X_{\phi(b_2)}\to Y_{b_0}$ and $f^{012}_{b_0b_1b_2}: I^{012}\times X_{\phi(b_2)}\to Y_{b_0}$ are illustrated by figure 2 (along the dotted line these two maps coincide)

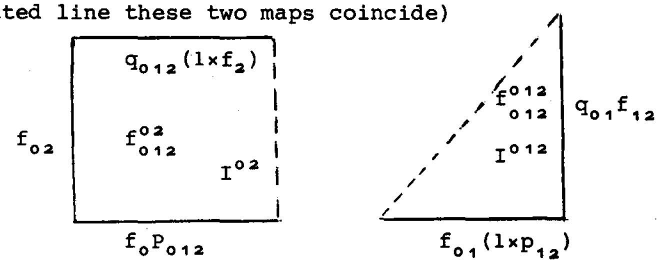

Fig. 2

Remark 1. This definition of a coherent map in fact is the same as the cubical-simplicial definition of a coherent map in [6] and [7] where instead of the space $I^{\underline{j}}$ appears $I^{n-k}\times\Delta^k$. Here instead of the simplex $\Delta^k=\{s=(s_0,\dots,s_k):s_0\geq 0,\dots,s_k\geq 0, s_0+\dots+s_k=1\}$ is used the subspace of the cube $\nabla^k=\{(t''_1,\dots,t''_k): 1\geq t''_1\geq\dots\geq t''_k\geq 0\}$.

From the definition of a coherent map in this paper to the earlier definition we can pass as in [8] by permutation of coordinates $t=(t_0,\dots,t_n)\to(t',s)$, $t'=(t_1,\dots,t_{n-k})\in I^{n-k}$ where

$$t'_{j-i} = t_j, \qquad j_i < j < j_{i+1}$$
$$s_i = t_{j_i}, \qquad 1 \le i < k$$

and ∇^k and Δ^k are naturally homeomorphic by the mapping given by

$$s_1 = t''_1 - t''_2, \dots, s_{k-1} = t''_{k-1} - t''_k, \; s_k = t''_k$$

To define the composition of coherent maps we decompose $I^{\underline{j}}$ into subpolyhedra defined by

$$K_i^{\underline{j}} = \{(t_1,\dots,t_n) : t_{j_i} \le \tfrac{1}{2} \le t_{j_{i+1}}\}, \; i=0,1,\dots,k \tag{3}$$

For n=2 the decomposition of I^{02} and I^{012} is shown on figure 3.

Let $f: \underline{X} \to \underline{Y}$ and $g: \underline{Y} \to \underline{Z} = (Z_c, r_{\underline{c}}, C)$ be coherent maps. Let g be given by the function ψ and maps $g_{\underline{c}}^{\underline{j}}: I^{\underline{j}} \times Y_{\psi(c_n)} \to Z_{c_o}$. Then the composition $h=gf: \underline{X} \to \underline{Z}$ is given by the function $\chi = \psi\phi$ and maps $h_{\underline{c}}^{\underline{j}}: I^{\underline{j}} \times X_{\phi\psi(c_n)} \to Z_{c_o}$ defined for n=0 with $h_{c_o} = g_{c_o} f_{\psi(c_o)}$ and for $n > 0$ and $(t_1,\dots,t_n) \in K_i^{\underline{j}}$ with

$$h_{\underline{c}}^{\underline{j}}(t_1,\dots,t_n,x) = g^{j_o\dots j_i}_{c_o\dots c_{j_i}}(t'_1,\dots,t'_{j_i}, f^{j_i-j_i\dots j_k-j_i}_{\phi(c_{j_i}\dots c_n)}(t'_{j_i+1},\dots,t'_n,x)) \tag{4}$$

where

$$t'_j = t_j, \qquad j_1 < j < j_{i+1}$$
$$t'_{j_\ell} = 2t_{j_\ell} - 1, \qquad 1 \le \ell \le i \tag{5}$$
$$t'_{j_\ell} = 2t_{j_\ell}, \qquad 1+i \le \ell \le k$$

In order to give the definition of coherent homotopy between two coherent maps, first for any strictly increasing sequence of integers $\underline{j} = (j_o,\dots,j_k)$, $0 = j_o < \dots < j_k = n$ we define two maps $u^{\underline{j}}, v^{\underline{j}}: I^{\underline{j}} \to I^n$ with $u^{\underline{j}}(t_1,\dots,t_n) = (t''_1,\dots,t''_n)$ where

$$t''_{j_i+1} = 1 - t_{j_{i+1}}$$
$$t''_{j+1} = t_j(1 - t_{j_{i+1}}), \quad j_i < j < j_{i+1} \tag{6}$$

and with $v^{\underline{j}}(t_1,\dots,t_n) = (t''_1,\dots,t''_n)$ where

$$t''_{j_{i+1}} = t_{j_{i+1}}$$
$$t''_{j} = t_j t_{j_{i+1}}, \quad j_i < j < j_{i+1} \tag{7}$$

Now, let $\underline{X}=(X_a, P_{\underline{a}}, A)$ and $\underline{Y}=(Y_b, q_{\underline{b}}, B)$ be coherent inverse systems. If $\phi: B \to A$ is a strictly increasing function, then the set $\phi(B) \subset A$ is a directed set. Let $\Phi: B \to A$ be a strictly increasing function such that if $\phi(b)=\phi(b')$ for $b,b' \in B$, then also $\Phi(b)=\Phi(b')$. Then by mapping $\phi(b)$ to $\Phi(b)$ it is defined a strictly increasing function $\phi(B) \to A$.

We define a coherent map $p(\phi,\Phi): \underline{X} \to (X_{\phi(b)}, P_{\phi(b_o \ldots b_n)}, \phi(B))$ given by maps $P^{j}_{\phi(\underline{b})}: I^{j} \times X_{\phi(b_n)} \to X_{\phi(b_o)}$ defined by

$$P^{j}_{\phi(b)}(t_o,\ldots,t_n,x) = p_{\phi(b_o \ldots b_{j_i})\Phi(b_{j_i} \ldots b_n)} v^{j_o \ldots j_i}(t'_1,\ldots,t_{j_i}) \, u^{j_i - j_i \ldots j_k - j_i}(t'_{j_i+1},\ldots,t'_n), x) \tag{8}$$

where $t'_1,\ldots,t'_n$ are defined as in the definition of the composition. After the computation we obtain

$$P^{j}_{\phi(\underline{b})}(t_1,\ldots,t_n,x) = p_{\phi(b_o \ldots b_{j_i})\Phi(b_{j_i} \ldots b_n)}(\tau_1,\ldots,\tau_n,x) \tag{8}$$

where for $0 \le \ell \le i-1$

$$\tau_{j_{\ell+1}} = 2t_{j_{\ell+1}} - 1 \tag{9}$$
$$\tau_j = t_j(2t_{j_{\ell+1}} - 1), \quad j_\ell < j < j_{j_{\ell+1}}$$

and for $i \le \ell \le k-1$

$$\tau_{j_\ell+1} = 1 - 2t_{j_{\ell+1}}$$
$$\tau_{j+1} = t_j(1-2t_{j_{\ell+1}}), \quad j_\ell < j < j_{j_{\ell+1}} \tag{10}$$

For $n=0$, $P_{\phi(b_o)}: X_{\Phi(b_o)} \to X_{\phi(b_o)}$, $P_{\phi(b_o)}(x) = p_{\phi(b_o)\Phi(b_o)}(x)$.
For $n=1$, a map $P_{\phi(b_o b_1)}: I \times X_{\Phi(b_1)} \to X_{\phi(b_o)}$ is given by

$$P_{\phi(b_0 b_1)}(t_1,x) = \begin{cases} P_{\phi(b_0 b_1)\Phi(b_1)}(2t_1-1,x), & t_1 \geq \frac{1}{2} \\ P_{\phi(b_0)\Phi(b_0 b_1)}(1-2t_1,x), & \frac{1}{2} \geq t_1 \end{cases} \tag{11}$$

For n=2 maps $P^{02}_{\phi(b_0 b_1 b_2)}: I^{02}\times X_{\Phi(b_2)} \to X_{\phi(b_0)}$ and $P^{012}_{\phi(b_0 b_1 b_2)}: I^{012}\times X_{\Phi(b_2)} \to X_{\phi(b_0)}$ are illustrated on figure 3

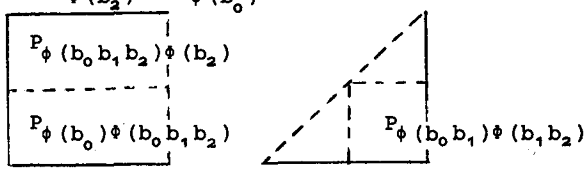

Fig. 3

Finally, let f,f': $\underline{X} \to \underline{Y}$ be coherent maps given by functions ϕ,ϕ' and maps $f^{j}_{\underline{b}}, f'^{j}_{\underline{b}}$ respectively.

Coherent maps f and f′ are coherently homotopic if

1) There exists a strictly increasing function $\Phi: B \to A$ such that $\Phi > \phi$, $\Phi > \phi'$ and $\Phi(b)=\Phi(b')$ if $\phi(b)=\phi(b')$ or $\phi'(b)=\phi'(b')$

2) There exists a coherent map $F: I\times\underline{X} \to \underline{Y}$, $(I\times\underline{X}=(I\times X_a, 1\times P_{\underline{a}}, A))$ given by the function Φ and maps $F^{j}_{\underline{b}}: I\times I^{j}\times X_{\Phi(b_n)} \to Y_{b_0}$ such that

$$\begin{aligned} F^{j}_{\underline{b}}(0,t,x) &= (f\circ P(\phi,\Phi))^{j}_{\underline{b}}(t,x) \\ F^{j}_{\underline{b}}(1,t,x) &= (f'\circ P(\phi',\Phi))^{j}_{\underline{b}}(t,x) \end{aligned} \tag{12}$$

Remark 2. Let coherent maps f,f': $\underline{X} \to \underline{Y}$ be defined by the same function ϕ and by maps $f^{j}_{\underline{b}}, f'^{j}_{\underline{b}}$ respectively. If there exists a coherent map $H: I\times\underline{X} \to \underline{Y}$ given by ϕ and maps $H^{j}_{\underline{b}}: I\times I^{j}\, X_{\phi(b_n)} \to Y_{b_0}$ such that

$$\begin{aligned} H^{j}_{\underline{b}}(0,t,x) &= f^{j}_{\underline{b}}(t,x) \\ H^{j}_{\underline{b}}(1,t,x) &= f'^{j}_{\underline{b}}(t,x) \end{aligned} \tag{13}$$

then f and f′ are coherently homotopic, because for an arbitrary function $\Phi > \phi$, the coherent map $F: I\times\underline{X} \to \underline{Y}$ defined by $F=H\circ P(\phi,\Phi)$ satisfies the formulas (12).

The coherent identity map $1_{\underline{Y}}: \underline{Y} \to \underline{Y}$ consists of the identity function 1_B and of the maps $1_{\underline{b}}^{\underline{j}}: I^{\underline{j}} \times Y_{b_n} \to Y_{b_o}$. For n=0 we have $1_{b_o} = 1_{Y_{b_o}}$ and for n > 0

$$1_{\underline{b}}^{\underline{j}}(t,x) = q_{\underline{b}}(t_1, \ldots, t_{j_1-1}, t_{j_1+1}, \ldots, t_{j_{k-1}-1}, 1, t_{j_{k-1}+1}, \ldots, t_{n-1}, x) \quad (14)$$

The category Coh has as objects coherent inverse systems, and the morphisms are coherent homotopy classes of coherent maps. The identity morphism is the coherent homotopy class of the identity map and the composition of morphisms is defined as the composition of homotopy classes.

<u>Remark 3</u>. If by transformation from Remark 1 coherent maps in category COH are expressed by the definition in this paper instead of the ealier definition then the definition of coherent homotopy of two coherent maps f,f′ in COH is: There exists a coherent map H given by function $\Phi > \phi, \phi'$ and maps $H_{\underline{b}}^{\underline{j}}: I \times I^{\underline{j}} \times X_{\phi(b_n)} \to Y_{b_o}$ such that

$$\begin{aligned} H_{\underline{b}}^{\underline{j}}(0,t,x) &= f_{\underline{b}}^{\underline{j}}(t, p_{\phi(b_n)\Phi(b_n)}(x)) \\ H_{\underline{b}}^{\underline{j}}(1,t,x) &= f'^{\underline{j}}_{\underline{b}}(t, p_{\phi'(b_n)\Phi(b_n)}(x)) \end{aligned} \quad (15)$$

We will show that definition of the coherent homotopy given in this paper is stronger. The proof of this statement is as follows:

If f,f′ are homotopic in Coh then there exists a homotopy F given by function and maps $F_{\underline{b}}^{\underline{j}}$ such that conditions 1) and 2) and specially formulas (12) from the definition of coherent homotopy hold. Then for $t=(t_1,\ldots,t_n) \in K_{\underline{k}}^{\underline{j}}$ from definition of map P we have

$$\begin{aligned} F_{\underline{b}}^{\underline{j}}(0,t,x) &= f_{\underline{b}}^{\underline{j}}(t'_1,\ldots,t'_n, p_{\phi(b_n)\Phi(b_n)}(x)) \\ F_{\underline{b}}^{\underline{j}}(1,\tau,t,x) &= f'^{\underline{j}}_{\underline{b}}(t'_1,\ldots,t'_n, p_{\phi'(b_n)\Phi(b_n)}(x)) \end{aligned} \quad (16)$$

where

$$t'_{j_i} = 1-2t_{j_i}, \qquad (17)$$
$$t'_j = t_j, \quad j_i < j < j_{i+1}$$

Now, we can define a coherent homotopy $H^j_{\underline{b}}: I\times I^j\times X_{\phi(b_n)} \to Y_{b_o}$ by putting

$$H^j_{\underline{b}}(s,t,x) = F^j_{\underline{b}}(s,t''_1,\ldots,t''_n,x) \qquad (18)$$

where

$$t''_{j_i} = \frac{1}{2}(1-t_{j_i}) \qquad (19)$$
$$t''_j = t_j, \quad j_i < j < j_{i+1}$$

Then for such defined coherent homotopy H and $t\in I^j$ formulas (15) hold.

Remark 4. The advantage of this new definition of a coherent homotopy is that maps which appear in formulas (12) are coherent, while maps in formulas (15) are coherent only in special casses.

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КОХЕРЕНТНА КАТЕГОРИЈА Кох

Никита Шекутковски

Р е з и м е

Конструирана е нова кохерентна категорија на кохерентните инверзни системи, која од порано дефинираната категорија во [6] и [7] се разликува во дефиницијата на кохерентна хомотопија. Добиената категорија со оваа посилна дефиниција на кохерентна хомотопија ги исполнува барањата на теоријата претставена без експлицитно зададени формули во [1] и [2].