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A CLASS OF UNIVALENT FUNCTION WITH FIXED FINITELY MANY COEFFICIENTS

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Abstract

In this paper a subclass of the class of univalent function is introduced by use of the Ruscheweuh derivative. Functions of this class have negative coefficients and have fixed finitely many coefficients. Necessary and sufficient conditions and extreme points for this class are provided. Also, it is shown that this classe is closed under convex linear combination and convolution and radius of starlikeness and convexity is found.

1. Introduction

Let $\mathcal{A}$ denote the family of functions

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

where $a_n \geq 0$ that are analytic in the unit disc $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ and let the subclass Ω consists of functions, f in $\mathcal{A}$ which are univalent in Δ. For $0 \leq \beta < 1, 0 \leq \alpha < 1$ and $\lambda = 0, 1, 2, \ldots$, we let $\Omega(\alpha, \beta, \lambda)$ consist of functions f in Ω satisfying the condition

$$Re\left\{\frac{z(D^\lambda f(z))'}{(1-\alpha)D^\lambda f(z) + \alpha z^2 (D^\lambda f(z))''}\right\} > \beta.$$

The operator $D^\lambda f$ is the Ruscheweyh derivative [4] of f defined by

$$D^\lambda f(z) = \frac{z(z^{\lambda-1}f(z))^{(\lambda)}}{\lambda!} = \frac{z}{(1-z)^{\lambda+1}} * f(z) = z - \sum_{n=2}^{\infty} B_n(\lambda) a_n z^n$$

where

$$B_n(\lambda) = \binom{n+\lambda-1}{\lambda} = \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)}{(n-1)!}.$$

Here the operation $*$ stands for the convolution of two power series $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ denoted by $(f*g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$.

Definition 1.1. [2]: *A convex function is one which maps the unit disk Δ conformally onto a convex domain, also a function $f(z) \in \mathcal{A}$ is said to be convex of order $\beta (0 \le \beta < 1)$ if and only if*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta, \qquad z \in \Delta$$

we use $k(\beta)$ for the class of convex function of order β and k for the class of convex function $k(0) = k$.

Definition 1.2. *We can say that a function is starlike if it is a conformal mapping of the unit disk onto a domain starlike with respect to the origin, and also a function $f(z) \in \mathcal{A}$ is said to be starlike of order $\beta(0 \le \beta < 1)$ if and only if*

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \qquad z \in \Delta.$$

We use notation $S^*(\beta)$ for the class of starlike functions of order β and S^* for the class of starlike function, $S^*(0) = S^*$. Note that $k \subset S^* \subset \mathcal{A}$ and Koebe function [2] is starlike but not convex.

Definition 1.3. *The class $\Omega_{c_k}(\alpha, \beta, \lambda) \subset \Omega(\alpha, \beta, \lambda)$ consists of functions of the form*

$$f(z) = z - \sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))c_i}{(\alpha\beta(1+i-i^2)+i-\beta)B_i(\lambda)} z^i - \sum_{n=k+1}^{\infty} a_n z^n, \qquad (2)$$

where $k = 2, 3, \ldots$, $a_n \ge 0$ for $n = k+1, k+2, \ldots$, $0 \le c_i \le 1$ for $i = 2, 3, \ldots, k$, and $0 \le \sum_{i=2}^{k} c_i \le 1$.

K. K. Dixit and I. B. Misra [1] have studied the special classes and was named $UCT(\alpha, c_k)$.

The class $\Omega(\alpha,\beta,\lambda)$ is of special interest because it contains many well known classes of univalent functions. In particular, for $\alpha = 0$ and $0 \le \lambda \le 1$ it provides a transition from starlike function to convex function. More precisely $\Omega(0,\beta,0)$ is the class of starlike function of order β and $\Omega(0,\beta,1)$ is the class of convex function of order β. S. Shams and S. R. Kulkarni introduced and studied the class $\Omega(\alpha,\beta,\lambda)$ in [3].

2. Main Results

We will need the following lemma whose details can be found in [5].

Lemma 2.1. *Let $f(z) \in \mathcal{A}$ be of form* (1). *Then $f(z) \in \Omega(\alpha,\beta,\lambda)$ if and only if*

$$\sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)} B_n(\lambda)a_n < 1$$

where $0 \le \alpha < 1$, $0 \le \beta < 1$ *and* $\lambda > -1$.

The following theorem gives a necessary and sufficient condition for a function to be in $\Omega_{c_k}(\alpha,\beta,\lambda)$.

Theorem 2.1. *Let $f(z) \in \mathcal{A}$ be defined by* (2). *Then f is in the class $\Omega_{c_k}(\alpha,\beta,\lambda)$ if and only if*

$$\sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)} B_n(\lambda)a_n < 1 - \sum_{i=2}^{k} c_i.$$

Proof. Letting

$$a_i = \frac{[1-\beta(1-\alpha)]c_i}{[\alpha\beta(1+i-i^2)+i-\beta]B_i(\lambda)}$$

we say that $f \in \Omega_{c_k}(\alpha,\beta,\lambda) \subset \Omega(\alpha,\beta,\lambda)$ if and only if

$$\sum_{i=2}^{k} \frac{\alpha\beta(1+i-i^2)+i-\beta}{1-\beta(1-\alpha)} a_i B_i(\lambda) + \sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)} a_n B_n(\lambda) < 1$$

or

$$\sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)} a_n B_n(\lambda) < 1 - \sum_{i=2}^{k} c_i$$

and the proof is complete. □

Corollary 2.1. *If the function defined by* (2) *is in* $\Omega_{c_k}(\alpha,\beta,\lambda)$ *then for any* $n \geq k+1$ *we have*

$$a_n \leq \frac{(1-\beta(1-\alpha))(1-\sum\limits_{i=2}^{k} c_i)}{[\alpha\beta(1+n-n^2)+n-\beta]B_n(\lambda)}.$$

This result is sharp due to the function $f(z)$ *defined by*

$$f(z) = z - \sum_{i=2}^{k} \frac{[1-\beta(1-\alpha)]c_i}{[\alpha\beta(1+i-i^2)+i-\beta]B_i(\lambda)} z_i -$$

$$- \frac{(1-\beta(1-\alpha))(1-\sum\limits_{i=2}^{k} c_i)}{[\alpha\beta(1+n-n^2)+n-\beta]B_n(\lambda)} z^n.$$

We now give some special cases of Theorem 2.1.

Corollary 2.2. *The function*

$$f(z) = z - \sum_{i=2}^{k} \frac{(1-\beta)i}{(i-\beta)} c_i z^i - \sum_{n=k+1}^{\infty} a_n z^n$$

is in $\Omega_{c_k}(0,\beta,0)$, *i.e.,* $f(z) \in S^*(\beta)$, *if and only if*

$$\sum_{n=k+1}^{\infty} \frac{n-\beta}{1-\beta} a_n < 1 - \sum_{i=2}^{k} c_i.$$

Corollary 2.3. *The function*

$$f(z) = z - \sum_{i=2}^{k} \frac{1-\beta}{i(i-\beta)} c_i z^i - \sum_{n=k+1}^{\infty} a_n z^n$$

is in $\Omega_{c_k}(0,\beta,1)$, *i.e.,* $f(z) \in K(\beta)$, *if and only if*

$$\sum_{n=k+1}^{\infty} \frac{n(n-\beta)}{1-\beta} a_n < 1 - \sum_{i=2}^{k} c_i.$$

We claim that all these results are entirely new.

Remark 2.1. *Since* $0 \leq \lambda_2 < \lambda_1$ *implies* $B_n(\lambda_2) \subset B_n(\lambda_1)$, *we note that* $\Omega_{c_k}(\alpha,\beta,\lambda_1) \subset \Omega_{c_k}(apha,\beta,\lambda_2)$.

Theorem 2.2. *Let for $j = 1, 2, \cdots, m$ the functions*

$$f_j(z) = z - \sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))c_i}{(\alpha\beta(1+i-i^2)+i-\beta)B_i(\lambda)} z^i - \sum_{n=k+1}^{\infty} a_{n,j} z^n \qquad (2.3)$$

be in $\Omega_{c_k}(\alpha, \beta, \lambda)$. *Then the function* $g(z) = \sum_{j=1}^{m} d_j f_j(z)$ *where* $\sum_{j=1}^{m} d_j = 1, 0 \le \sum_{i=2}^{k} c_i \le 1, 0 \le c_i \le 1$ *is in* $\Omega_{c_k}(\alpha, \beta, \lambda)$.

Proof. For every $j \in \{1, 2, \cdots, m\}$ by making use of Theorem 2.1 we have

$$\sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)} B_n(\lambda) a_{n,j} < 1 - \sum_{n=2}^{k} c_i.$$

But

$$g(z) = z - \sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))c_i}{[\alpha\beta(1+i-i^2)+i-\beta]B_i(\lambda)} z^i - \sum_{n=k+1}^{\infty} (\sum_{j=1}^{m} d_j a_{n,j}) z^n.$$

Consequently we obtain

$$\sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)} B_n(\lambda) (\sum_{j=1}^{m} d_j a_{n,j}) =$$

$$= \sum_{j=1}^{m} \sum_{n=k+1}^{\infty} \left(\frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)} B_n(\lambda) a_{n,j} \right) d_j <$$

$$< \sum_{j=1}^{m} \left(1 - \sum_{i=2}^{k} c_i \right) d_j =$$

$$= 1 - \sum_{i=2}^{k} c_i$$

and the proof is complete. □

Remark 2.2. *If* $f_1(z), f_2(z)$ *are in* $\Omega_{c_k}(\alpha, \beta, \lambda)$ *then the function* $g(z)$ *defined by* $g(z) = \frac{1}{2}(f_1(z) + f_2(z))$ *is also in* $\Omega_{c_k}(\alpha, \beta, \lambda)$.

Corollary 2.4. *The class* $\Omega_{c_k}(\alpha, \beta, \lambda)$ *is closed under convex linear combination, i.e., this class is a convex set.*

Theorem 2.3. *Let the function $f_j(z)$ defined by (2.3) be in $\Omega_{c_k}(\alpha,\beta,\lambda)$ for each $j = 1, 2, \cdots, m$, then the function*

$$h(z) = z - \sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))c_i}{(\alpha\beta(1+i-i^2)+i-\beta)B_i(\lambda)} z^i - \sum_{n=k+1}^{\infty} b_n z^n, \quad (b_n \geq 0)$$

is also in $\Omega_{c_k}(\alpha,\beta,\lambda)$, where $b_n = \frac{1}{m}\sum_{j=1}^{m} a_{n,j}$.

Proof. We must show that

$$\sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)} B_n(\lambda) b_n < 1 - \sum_{i=2}^{k} c_i.$$

But we have

$$\sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)} B_n(\lambda) b_n =$$

$$= \sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)} B_n(\lambda) \left(\frac{1}{m}\sum_{j=1}^{m} a_{n,j}\right) =$$

$$= \frac{1}{m}\sum_{j=1}^{m}\left[\sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)} B_n(\lambda) a_{n,j}\right] <$$

$$< \frac{1}{m}\sum_{j=1}^{m}\left(1 - \sum_{i=2}^{k} c_i\right) =$$

$$= 1 - \sum_{i=2}^{k} c_i,$$

and the proof is complete. □

In order to determine the extreme points of the class $\Omega_{c_k}(\alpha,\beta,\lambda)$ first we will prove the following theorem.

Theorem 2.4. *A function $f(z)$ is in $\Omega_{c_k}(\alpha,\beta,\lambda)$ if and only if it can be expressed in the form*

$$f(z) = \sum_{n=k}^{\infty} \mu_n f_n(z)$$

where $\mu_n \geq 0 (n \geq k)$, $\sum\limits_{n=k}^{\infty} \mu_n = 1$,

$$f_k(z) = z - \sum_{i=2}^{k} \frac{[1-\beta(1-\alpha)]c_i}{[\alpha\beta(1+i-i^2)+i-\beta]B_i(\lambda)} z^i \tag{3}$$

and

$$f_n(z) = z - \sum_{i=2}^{k} \frac{[1-\beta(1-\alpha)]c_i}{[\alpha\beta(1+i-i^2)+i-\beta]B_i(\lambda)} z^i -$$

$$- \frac{(1-\beta(1-\alpha))(1-\sum\limits_{i=2}^{k} c_i)}{[\alpha\beta(1+n-n^2)+n-\beta]B_n(\lambda)},$$

for $n = k+1, k+2, \ldots$.

Proof. Suppose that $f(z)$ can be expressed in the form $f(z) = \sum\limits_{n=k}^{\infty} \mu_n f_n(z)$. Then we have

$$f(z) = \mu_k f_k(z) + \sum_{n=k+1}^{\infty} \mu_n f_n(z) =$$

$$= \mu_k z - \mu_k \sum_{i=2}^{\infty} \frac{(1-\beta(1-\alpha))c_i}{[\alpha\beta(1+i-i^2)+i-\beta]B_i(\lambda)} z^i + \sum_{n=k+1}^{\infty} \mu_n z -$$

$$- \sum_{n=k+1}^{\infty} \mu_n \left(\sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))c_i}{[\alpha\beta(1+i-i^2)+i-\beta]B_i(\lambda)} z^i \right) -$$

$$- \sum_{n=k+1}^{\infty} \mu_n \left(\frac{(1-\beta(1-\alpha))(1-\sum\limits_{i=2}^{k} c_i)}{[\alpha\beta(1+n-n^2)+n-\beta]B_n(\lambda)} \right) =$$

$$= \left(\mu_k + \sum_{n=k+1}^{\infty} \mu_n \right) z - \left(\mu_k + \sum_{n=k+1}^{\infty} \mu_n \right) \sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))c_i}{[\alpha\beta(1+i-i^2)+i-\beta]} z^i -$$

$$- \sum_{n=k+1}^{\infty} \frac{(1-\beta(1-\alpha))(1-\sum\limits_{i=2}^{k} c_i)}{[\alpha\beta(1+n-n^2)+n-\beta]B_n(\lambda)} z^n =$$

$$=z-\sum_{i=2}^{k}\frac{(1-\beta(1-\alpha))c_i}{[\alpha\beta(1+i-i^2)+i-\beta]B_i(\lambda)}z^i-$$

$$-\sum_{n=k+1}^{\infty}\frac{\mu_n(1-\beta(1-\alpha))(1--\sum_{i=2}^{k}c_i)}{[\alpha\beta(1+n-n^2)+n-\beta]B_n(\lambda)}z^n.$$

Finally we can write

$$\sum_{n=k+1}^{\infty}\frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)}\cdot\frac{\mu_n(1-\beta(1-\alpha))(1-\sum_{i=2}^{k}c_i)}{[\alpha\beta(1+n-n^2)+n-\beta]B_n(\lambda)}B_n(\lambda)=$$

$$=(1-\sum_{i=2}^{k}c_i)\sum_{n=k+1}^{\infty}\mu_n=(1-\sum_{i=2}^{k}c_i)(1-\mu_k)<1-\sum_{i=2}^{k}c_i.$$

Therefore $f\in\Omega_{c_k}(\alpha,\beta,\lambda)$.

Conversely, let $f\in\Omega_{c_k}(\alpha,\beta,\lambda)$, that is

$$f(z)=z-\sum_{i=2}^{k}\frac{(1-\beta(1-\alpha))c_i}{[\alpha\beta(1+i-i^2)+i-\beta]B_i(\lambda)}z^i-\sum_{n=k+1}^{\infty}a_nz^n.$$

We may set

$$\mu_n=\frac{[\alpha\beta(1+n-n^2)+n-\beta]B_n(\lambda)}{(1-\beta(1-\alpha))(1-\sum_{i=2}^{k}c_i)}a_n,\quad (n\geq k+1).$$

Therefore $\mu_n\geq 0$ and if we set $\mu_k=1-\sum_{n=k+1}^{\infty}\mu_n$ then we have

$$f(z)=z-\sum_{i=2}^{k}\frac{(1-\beta(1-\alpha))c_i}{(\alpha\beta(1+i-i^2)+i-\beta)B_i(\lambda)}z^i-$$

$$-\sum_{n=k+1}^{\infty}\frac{(1-\beta(1-\alpha))(1-\sum_{i=2}^{k}c_i)}{(\alpha\beta(1+n-n^2)+n-\beta)B_n(\lambda)}\mu_nz^n=$$

$$=f_k(z) - \sum_{n=k+1}^{\infty} \left(z - \sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))c_i}{(\alpha\beta(1+i-i^2)+i-\beta)B_i(\lambda)} z^i - f_n(z) \right) \mu_n =$$

$$=f_k(z) - \sum_{n=k+1}^{\infty} (f_k(z) - f_n(z))\mu_n =$$

$$=(1 - \sum_{n=k+1}^{\infty} \mu_n) f_k(z) + \sum_{n=k+1}^{\infty} \mu_n f_n(z) = \sum_{n=k}^{\infty} \mu_n f_n(z).$$

□

Corollary 2.5. *The extreme points of the class* $\Omega_{c_k}(\alpha, \beta, \lambda)$ *are the function* $f_n(z) (n \geq k)$ *given by* (3) *and* (4).

Theorem 2.5. *Let* $f \in \Omega_{c_k}(\alpha, \beta, \lambda)$ *and*

$$d_i = \frac{(1-\beta(1-\alpha))c_i}{|\alpha\beta(1+i-i^2)+i-\beta|B_i(\lambda)} \quad (2 \leq i \leq k). \tag{4}$$

Then the function

$$g(z) = z - \sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))d_i}{(\alpha\beta(1+i-i^2)+i-\beta)B_i(\lambda)} z^i - \sum_{n=k+1}^{\infty} a_n z^n$$

is in the class $\Omega_{d_k}(\alpha, \beta, \lambda)$.

Proof. It can be verified that $|\alpha\beta(1+i-i^2)+i-\beta|B_i(\lambda) > 1$, $i = 2, 3, \ldots, k$. Therefore

$$0 \leq d_i = \frac{(1-\beta(1-\alpha))c_i}{|\alpha\beta(1+i-i^2)+i-\beta|B_i(\lambda)} < c_i \leq 1$$

and

$$0 \leq \sum_{i=2}^{k} d_i < \sum_{i=2}^{k} c_i \leq 1.$$

Now we can write

$$\sum_{n=k+1}^{\infty} \frac{(\alpha\beta(1+i-i^2)+i-\beta)}{(1-\beta(1-\alpha))(1-\sum_{i=2}^{k} d_i)} B_n(\lambda) a_n <$$

$$< \sum_{n=k+1}^{\infty} \frac{(\alpha\beta(1+i-i^2)+i-\beta)}{(1-\beta(1-\alpha))(1-\sum_{i=2}^{k} c_i)} B_n(\lambda) a_n < 1$$

and this completes the proof. □

Theorem 2.6. *Suppose that $f, g \in \Omega_{c_k}(\alpha,\beta,\lambda)$. Then*

$$(f*g)(z) = z - \sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))^2 c_i^2}{[\alpha\beta(1+i-i^2)+i-\beta]^2 (B_i(\lambda))^2} z^i - \sum_{n=k+1}^{\infty} a_n b_n z^n$$

is in $\Omega_{d_k}(\alpha,\beta,\lambda_1)$ if

$$B_n(\lambda_1) < \frac{(B_n(\lambda))^2}{1-\sum\limits_{i=2}^{k} d_i},$$

where d_i, $i = 2, 3, \ldots, k$, are given by (4).

Proof. By making use of (4) we can write

$$(f*g)(z) = z - \sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))d_i}{|\alpha\beta(1+i-i^2)+i-\beta| B_i(\lambda)} - \sum_{n=k+1}^{\infty} a_n b_n z^n$$

and by Theorem 2.1 we have

$$\sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{(1-\beta(1-\alpha))(1-\sum\limits_{i=2}^{k} d_i)} a_n B_n(\lambda) < 1$$

and

$$\sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{(1-\beta(1-\alpha))(1-\sum\limits_{i=2}^{k} d_i)} b_n B_n(\lambda) < 1.$$

By applying the Cauchy-Schwarz inequality we have

$$\sum_{n=k+1}^{\infty} \frac{|\alpha\beta(1+n-n^2)+n-\beta|}{(1-\beta(1-\alpha))(1-\sum\limits_{i=2}^{k} d_i)} \sqrt{a_n b_n} B_n(\lambda) < 1. \tag{5}$$

We want to prove that

$$\sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{(1-\beta(1-\alpha))(1-\sum\limits_{i=2}^{k} d_i)} a_n b_n B_n(\lambda_1) < 1. \tag{6}$$

Therefore, in view of (5) the inequality (6) holds if

$$\sqrt{a_n b_n}\frac{B_n(\lambda_1)}{B_n(\lambda)} < 1,$$

i.e.,

$$B_n(\lambda_1) < \frac{(B_n(\lambda))^2}{1 - \sum\limits_{i=2}^{k} d_i}.$$

□

Using Remark 2.1 and Theorem 2.6 we receive the following corollary.

Corollary 2.6. *Let* $\lambda_1 \neq \lambda_2$, $f(z) \in \Omega_{c_k}(\alpha, \beta, \lambda_1)$ *and* $g(z) \in \Omega_{c_k}(\alpha, \beta, \lambda_2)$.

i) *If* $\lambda_2 < \lambda_1$ *then* $f * g(z) \in \Omega_{c_k}(\alpha, \beta, \lambda)$ *where* $B_n(\lambda) < (B_n(\lambda_2))^2 / \left(1 - \sum\limits_{i=2}^{k} d_i\right)$.

ii) *If* $\lambda_1 < \lambda_2$ *then* $f * g(z) \in \Omega_{c_k}(\alpha, \beta, \lambda)$ *where* $B_n(\lambda) < (B_n(\lambda_1))^2 / \left(1 - \sum\limits_{i=2}^{k} d_i\right)$.

3. Important Properties of $\Omega_{c_k}(\alpha, \beta, \lambda)$

Definicija 3.1. *Let* $f(z)$ *and* $g(z)$ *be in the class* $\Omega_{c_k}(\alpha, \beta, \lambda)$, *then weighted mean* $h_j(z)$ *of* $f(z)$ *and* $g(z)$ *is given by*

$$h_j(z) = \frac{1}{2}[(1-j)f(z) + (1+j)g(z)].$$

Theorem 3.1. *If* $f(z)$ *and* $g(z)$ *are in the class* $\Omega_{c_k}(\alpha, \beta, \lambda)$, *then the weighted mean of* $f(z)$ *and* $g(z)$ *is also in* $\Omega_{c_k}(\alpha, \beta, \lambda)$.

Proof. By using Definition 3.1 we obtain

$$h_j(z) = \frac{1}{2}\left[(1-j)\left(z - \sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))c_i}{(\alpha\beta(1+i-i^2)+i-\beta)B_i(\lambda)} z^i - \sum_{n=k+1}^{\infty} a_n z^n\right) + \right.$$

$$\left. +(1+j)\left(z - \sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))c_i}{(\alpha\beta(1+i-i^2)+i-\beta)B_i(\lambda)} z^i - \sum_{n=k+1}^{\infty} b_n z^n\right)\right] =$$

$$= z - \sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))c_i}{(\alpha\beta(1+i-i^2)+i-\beta)B_i(\lambda)} z^i -$$

$$- \sum_{n=k+1}^{\infty} \frac{1}{2}((1-j)a_n + (1+j)b_n)z^n.$$

Since $f(z)$ and $g(z)$ are in the class $\Omega_{c_k}(\alpha,\beta,\lambda)$ using Theorem 2.1 we have

$$\sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)} B_n(\lambda)\frac{1}{2}((1-j)a_n + (1+j)b_n) =$$

$$= \frac{1}{2} \sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)}(1-j)B_n(\lambda)a_n +$$

$$+ \frac{1}{2} \sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)}(1+j)B_n(\lambda)b_n \le$$

$$\le \frac{1}{2}(1-j)\left(1-\sum_{i=2}^{k} c_i\right) + \frac{1}{2}(1+j)\left(1-\sum_{i=2}^{k} c_i\right) =$$

$$= 1 - \sum_{i=2}^{k} c_i,$$

and again by Theorem 2.1, $h_j(z) \in \Omega_{c_k}(\alpha,\beta,\lambda)$. □

Theorem 3.2. *Let functions $f_1(z), f_2(z), \cdots, f_m(z)$ defined by*

$$f_j(z) = z - \sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))c_i}{(\alpha\beta(1+i-i^2)+i-\beta)B_i(\lambda)} z^i - \sum_{n=k+1}^{\infty} a_{n,j} z^n,$$

$j = 1, 2, \ldots, m$, be in the class $\Omega_{c_k}(\alpha,\beta,\lambda)$. Then their arithmetic mean defined by $h(z) = \frac{1}{m}\sum_{j=1}^{\infty} f_j(z)$ is also in the class $\Omega_{c_k}(\alpha,\beta,\lambda)$.

Proof. One can verify that

$$h(z) = \frac{1}{m}\sum_{j=1}^{m}\left(z - \sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))c_i}{(\alpha\beta(1+i-i^2)+i-\beta)B_i(\lambda)} z^i - \sum_{n=k+1}^{\infty} a_{n,j} z^n\right) =$$

$$= z - \sum_{i=2}^{k} \frac{(1-\beta(1-\alpha))c_i}{(\alpha\beta(1+i-i^2)+i-\beta)B_i(\lambda)} z^i - \sum_{n=k+1}^{\infty}\left(\frac{1}{m}\sum_{j=1}^{m} a_{n,j}\right) z^n.$$

Since $f_j(z) \in \Omega_{c_k}(\alpha,\beta,\lambda)$ for every $j = 1, 2, \cdots, m$, from Theorem 2.1, we obtain

$$\sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)} B_n(\lambda) \left(\frac{1}{m}\sum_{j=1}^{m} a_{n,j}\right) \leq$$

$$\leq \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=k+1}^{\infty} \frac{\alpha\beta(1+n-n^2)+n-\beta}{1-\beta(1-\alpha)} B_n(\lambda) a_{n,j}\right) \leq$$

$$\leq \frac{1}{m}\sum_{j=1}^{m}\left(1-\sum_{i=2}^{k} c_i\right) = 1-\sum_{i=2}^{k} c_i,$$

which in view of Theorem 2.1 gives the conclusion of this theorem. □

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КЛАСА НА ЕДНОЛИСНИ ФУНКЦИИ СО КОНЕЧНО МНОГУ ФИКСИРАНИ КОЕФИЦИЕНТИ

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Резиме

Во овој труд со помош на изводот на Рушевај (Ruscheweuh) дефинирана е поткласа на класата на еднолисни функции. Функциите од оваа класа имаат негативни коефициенти и имаат фиксирани конечен број од коефициентите. Дадени се потребни и доволни услови, како и екстремални точки за оваа класа. Исто така, покажано е дека оваа класа е затворена во однос на конвексната линеарна комбинација и во однос на конволуцијата. Пронајден е и радиусот на ѕвездоликост и на конвексност.

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