ANALOGY OF ONE THEOREM OF M. PETROVITCH

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1. M. Petrovitch has proved the following theorem: for the Riccati's integrable equation [1]

$$(1) \qquad y' + y^2 = f(x),$$

it may be attached a series of functions $\lambda_i(x)$, $i = 1, 2, \ldots$ of such kind, that every equation

$$y' + y^2 = f(x) + \lambda_i(x),$$

is integrable too, without any additional quadratures. His proof is based on a series of three successive transformations applied to equation (1) Modifying Petrovitch's proof, we shall show how can a similar result be deduced and how it can be generalised.

2. Suppose that in the equation (1) we replace the function y by*

$$y = \Delta_1(u) = \frac{u'}{u}, \quad u' = \frac{du}{dx}.$$

We have

$$(2) \qquad \Delta_2(u) = f(x)$$

With a new substitution

$$u' = \sqrt{f(x)}\, u_1,$$

we obtain

$$\Delta_2(u_1) = f_1(x),$$

where

$$f_1(x) = f + \frac{3}{4}\Delta_1^2(f) - \frac{1}{2}\Delta_2(f).$$

Exchanging successively the functions by

$$u'_{k-1} = \sqrt{f_{k-1}(x)}\, u_k,$$

we find

$$(3) \qquad \Delta_2(u_n) = f_n(x),$$

where

$$f_n(x) = f_{n-1} + \frac{3}{4}\Delta_1^2(f_{n-1}) - \frac{1}{2}\Delta_2(f_{n-1})$$

$$= f + \frac{1}{4}[\Delta_1^2(f) + \Delta_1^2(f_1) + \text{— — —} + \Delta_1^2(f_{n-1})] - \frac{1}{2}\Delta_1'(ff_1 \text{- - -} f_{n-1}).$$

On replacing y_n by $y_n = \Delta_1(u_n)$, the equation (3) is the Riccati's one

$$y_n' + y_u^2 = f(x) + \lambda_n(x)$$

where

$$\lambda_n(x) = \frac{1}{4}[\Delta_1^2(f) + \Delta_1^2(f_1) + \ldots + \Delta_1^2(f_{n-1})] - \frac{1}{2}\Delta_1'(ff_1 \ldots f_{n-1}).$$

It is obvious that the solutions of the equations (1) can be obtained of without any additional quadratures, because they can be expresed by the relation of

* We profit the relative derivatives as folows:

$$\Delta_1(u) = \frac{u'}{u}, \quad \Delta_2(u) = \frac{u''}{u}, \quad \Delta_2(u) = \Delta_1'(u) + \Delta_1^2(u)$$

$$y_n = y_{n-1} + \Delta_1\left(\frac{y_{n-1}}{\sqrt{f_{n-1}}}\right).$$

We notice that the given theorem above can be explained in the following way: the equation (1) is integrable, if one of the transformed equations (3) is of such sort.

3. If we make in equation (2) the transformation

$$u' = f u_1,$$

we have

$$\Delta_2 (u_1) + \Delta_1 (f) \Delta_1 (u_1) = \varphi_1,$$

where

$$\varphi_1 = f - \Delta_1' (f).$$

By the substitution

$$u_1' = \varphi_1 u_2,$$

we obtain

$$\Delta_2 (u_2) + \Delta_1 (\varphi\varphi_1) \Delta_1 (u_2) = \varphi_2, \ \varphi = f,$$

where

$$\varphi_2 = \varphi_1 - \Delta_1' (\varphi\varphi_1).$$

Continuing in the same way, we get by $u_{n-1}' = \varphi_{n-1} u_n$ the e quation

$$\Delta_2 (u_n) + \Delta_1 (\varphi\varphi_1 \dots \varphi_{n-1}) \Delta_1 (u_n) = \varphi_n$$

where

$$\varphi_n = \varphi_{n-1} - \Delta_1' (\varphi\varphi_1 \dots \varphi_{n-1})$$
$$= \varphi - \Delta_1' (\varphi^n \varphi_1^{n-1} \dots \varphi_{n-1}).$$

The corresponding Riccati's equation is

$$v' + v^2 = \Phi_n$$

with

$$\Phi_n = \varphi_{n-1} - \Delta_1' (\varphi \dots \varphi_{n-1}) + \frac{1}{4} \Delta_1^2 (\varphi\varphi_1 \dots \varphi_{n-1})$$

$$= \varphi - \frac{5}{4} \Delta_1' (\varphi \dots \varphi_{n-1}) + \frac{1}{4} \Delta_2 (\varphi \dots \varphi_{n-1}).$$

Thus we come to the analogical theorem: if the equation (1) is integrable, then there is a series of functions $\mu_1 (x)$ such, that the equations

$$y_i' + y_i^2 = f(x) + \mu_i (x),$$

with

$$\mu_i = \frac{1}{4} \Delta_2 (\varphi \dots \varphi_{n-1}) - \frac{5}{4} \Delta_1' (\varphi \dots \varphi_{n-i}),$$

are also integrable without any additional quadratures.

4. The general transformation $u' = f^k u_1$, k is real, includes both bofore mentoined cases.

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[1] *M. Petrovitch*, Theoreme sur l'equation de Riccati, Publications mathematiques de l'Universite de Belgrade, IV, p. 169.