ON CERTAIN RELATIONS FOR ORTHOGONAL CLASSICAL POLYNOMIALS

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The purpose of this note is to establish some properties of certain classes of special functions by an elementary method readily applicable to other classes of functions.

1. It is known that for two arbitrary functions $u=u(x)$, $v=v(x)$, n-times differentiable, there exists the adjoint Leibnitz formula [1]

$$u D^n v=\sum_{k=0}^{n}(-1)^k \binom{n}{k} D^{n-k}(vD^k u), \quad D=\frac{d}{dx}, \tag{1}$$

which we consider as an operational relation.

If we take

$$u=(x-1)^{-\beta}, \quad v=(-1)^{\alpha+n}(x-1)^{\beta+n},$$

where α and β are arbitrary parameters with $\alpha>-1$, $\beta>-1$, we obtain from (1)

$$(x-1)^{-\beta} D^n (x-1)^{\alpha+n}(x+1)^{\beta+n}= \tag{2}$$

$$=\sum_{k=0}^{n}\binom{n}{k}(\beta)_k D^{n-k}(x-1)^{\alpha+n}(x+1)^{n-k},$$

$$(\beta)_k=\beta(\beta+1)\ldots(\beta+k-1), \quad (\beta)_o=1.$$

By the definition of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ we have [2]

$$D^n(x-1)^{\alpha+n}(x+1)^{\beta+n}=2^n n!\,(x-1)^{\alpha}(x+1)^{\beta} P_n^{(\alpha,\beta)}(x),$$

so that (2) gives

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\frac{(\beta)_k}{2^k\, k!}(x-1)^k P_{n-k}^{(\alpha+k,0)}(x). \tag{3}$$

Using the well-known relation

$${}_2F_1(-n,\ a,\ b,\ x)=(-1)^n\frac{(a)_n}{(b)_n}x^n {}_2F_1(-n,\ 1-b-n,\ 1-a-n,\ x^{-1})$$

it follows from (3) after replacing α by $\alpha+\beta$ the desired result

$$P_n^{(\alpha,\ \beta)}(x)=\sum_{k=0}^{n}(-1)^k\frac{(\beta)_k}{k!}\ P_{n-k}^{(\alpha+\beta+k,\ 0)}(x)$$

The analogous formula

$$P_n^{(\alpha,\ \beta)}(x)=\sum_{k=0}^{n}\frac{(\beta)_k}{k!}\ P_{n-k}^{(\gamma,\ \alpha+\beta-\gamma+k)}(x).$$

given by Feldheim [3] may be obtained by the above method, if we take

$$u=(x-1)^{\gamma-\alpha}, \quad v=(x-1)^{\alpha+n}(x+1)^{-\alpha-n-\beta-2\gamma-1}.$$

2. The use of relation (1) can be profitably illustrated also by obtaining the so-called inverse formula or expansion of x^n in a series of orthogonal or other functions [4].

In fact, if we put in (1)

$$u=x^{n+\alpha}, \quad v=e^{-x},$$

after changing the order of the derivatives in the right side of (1) we obtain

$$(4) \qquad (-1)^n\, x^{n+\alpha}\, e^{-x}=\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\binom{n+\alpha}{n-k}(n-k)!\; D^k\, e^{-x}\, x^{\alpha+k}$$

But

$$D^n\, e^{-x}\, x^{\alpha+k}=e^{-x}\, x^{\alpha}\; k!\; L_k^{(\alpha)}(x),$$

where $L_n^{(\alpha)}(x)$ are the Laguerre polynomials.

Finally, then from (3) we have

$$\frac{x^n}{n!}=\sum_{k=o}^{n}(-1)^k\binom{n+\alpha}{n-k}L_k^{(\alpha)}(x).$$

REFERENCES

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[2] *E. Rainville*, Special Functions, The Macmilan Comp. New-York, 1967.

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[4] *D. W. Suda*, American Math. Monthly, 75 (1968), 643—644.

РЕЗИМЕ

НЕКОИ РЕЛАЦИИ ЗА КЛАСИЧНИТЕ ОРТОГОНАЛНИ ПОЛИНОМИ

Во трудот се даваат некои својства за извесни класи специјални функции со еден елементарен метод што згодно може да се примени и за други класи функции.