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INFINITESIMAL BENDING OF CURVES

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Abstract

In this paper we consider infinitesimal bending of curves in $\mathcal{R}^3$. The field of infinitesimal bending and deformed curve is given. Infinitesimal bending of circle is investigated. Variation of the torsion and curvature for a circle under infinitesimal bending is considered. Infinitesimal bending that plane curve remains in the plane is given. Infinitesimal bending is studied not only analytically, but also by drawing curves.

0. Introduction

Curve C_ε is infinitesimal deformation of a curve C if it is included in continuous family of curves

$$C_\varepsilon : \overline{r}_\varepsilon = \overline{r}_\varepsilon(u) , \quad u \in \mathcal{R} , \quad \varepsilon \in [0,1), \quad \varepsilon \to 0 ,$$

$$\overline{r}_\varepsilon : R \times [0,1) \to R^3 .$$

We get curve C for $\varepsilon = 0$. Giving different conditions, we get different types of deformations of curves. We will here consider infinitesimal bending of curves.

More information about infinitesimal bending of curves and surfaces one can get from [1], [2], [6], [4], [5].

Definition 0.1. *Let us consider a continuous regular curve*

$$C : \overline{r} = \overline{r}(u) , \tag{0.1}$$

included in a family of the curves

$$C_\varepsilon : \overline{r}_\varepsilon = \overline{r}(u) + \varepsilon\, \overline{z}(u), \quad (\varepsilon \geq 0, \quad \varepsilon \to 0, \quad \varepsilon \in \mathcal{R}) \tag{0.2}$$

where u is a real parameter, and *we get C* for $\varepsilon = 0$, $C = C_0$. *Family of curves C_ε is* **infinitesimal bending of the curve** *C if*

$$ds_\varepsilon^2 - ds^2 = o(\varepsilon), \tag{0.3}$$

where $\overline{z} = \overline{z}(u)$ is **infinitesimal bending field** *of the curve C.*

Definition 0.2. *Let C_ε be an infinitesimal bending of a curve C. Let $\mathcal{A}(u)$ be some a geometric magnitude of a curve C, and $\mathcal{A}_\varepsilon(u)$ corresponding magnitude of a curve C_ε and let*

$$\Delta\mathcal{A} = \mathcal{A}_\varepsilon - \mathcal{A} = \varepsilon\delta\mathcal{A} + \varepsilon^2\delta^2\mathcal{A} + \cdots + \varepsilon^n\delta^n\mathcal{A} + \cdots \tag{0.4}$$

Coefficients $\delta\mathcal{A}, \delta^2\mathcal{A}, \ldots, \delta^n\mathcal{A}, \ldots$ are the first, the second, n-th variation *of the geometric magnitude $\mathcal{A}$, respectively, under infinitesimal bending of the curve C.*

We will further on consider the first variation and we will simply call it variation. According to (0.4) we have

$$\delta\mathcal{A} = \lim_{\varepsilon \to 0} \frac{\Delta\mathcal{A}}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\mathcal{A}_\varepsilon - \mathcal{A}}{\varepsilon}$$

or

$$\delta\mathcal{A} = \frac{\partial\mathcal{A}}{\partial\varepsilon}\Big|_{\varepsilon=0}. \tag{0.5}$$

Theorem 0.1. [2] *Necessary and sufficient condition for $\overline{z}(u)$ to be an infinitesimal bending field of a curve C is to be*

$$d\overline{r}\, d\overline{z} = 0. \quad \square \tag{0.6}$$

1. Infinitesimal bending of a curve in $\mathcal{R}^3$

The next theorem is related to determination of the infinitesimal bending field of a curve C.

Theorem 1.1. *Infinitesimal bending field for curve C* (0.1) *is*

$$\overline{z}(u) = \int [p(u)\,\overline{n}(u) + q(u)\,\overline{b}(u)]\, du, \tag{1.1}$$

where $p(u)$, $q(u)$, are arbitrary integrable functions, and vectors $\overline{n}(u)$, $\overline{b}(u)$ are unit principal normal and binormal vector field of a curve C.

Proof. Since

$$d\overline{r} = \dot{\overline{r}}(u)\,du\,, \quad d\overline{z} = \dot{\overline{z}}(u)\,du\,,$$

according to (0.6) for infinitesimal bending field of a curve C we have

$$\dot{\overline{r}} \cdot \dot{\overline{z}} = 0\,, \quad \text{i.e.} \quad \dot{\overline{z}} \perp \dot{\overline{r}}\,. \tag{1.2}$$

Based on that we conclude that $\dot{\overline{z}}$ lies in the normal plane of the curve C, i.e.

$$\dot{\overline{z}}(u) = p(u)\,\overline{n}(u) + q(u)\,\overline{b}(u)\,, \tag{1.3}$$

where $p(u)$, $q(u)$ are arbitrary integrable functions. Integrating (1.3) we have (1.1). Since

$$\overline{b} = \frac{\dot{\overline{r}} \times \ddot{\overline{r}}}{|\dot{\overline{r}} \times \ddot{\overline{r}}|}\,, \quad \overline{n} = \frac{(\dot{\overline{r}} \cdot \dot{\overline{r}})\ddot{\overline{r}} - (\dot{\overline{r}} \cdot \ddot{\overline{r}})\dot{\overline{r}}}{|\dot{\overline{r}}|\,|\dot{\overline{r}} \times \ddot{\overline{r}}|} \tag{1.4}$$

infinitesimal bending field can be written in the form

$$\overline{z}(u) = \int \Big[p(u)\,\frac{(\dot{\overline{r}} \cdot \dot{\overline{r}})\ddot{\overline{r}} - (\dot{\overline{r}} \cdot \ddot{\overline{r}})\dot{\overline{r}}}{|\dot{\overline{r}}|\,|\dot{\overline{r}} \times \ddot{\overline{r}}|} + q(u)\,\frac{\dot{\overline{r}} \times \ddot{\overline{r}}}{|\dot{\overline{r}} \times \ddot{\overline{r}}|}\Big]du$$

where $p(u)$, $q(u)$ are arbitrary functions, or in the form

$$\overline{z}(u) = \int [P_1(u)\dot{\overline{r}} + P_2(u)\ddot{\overline{r}} + Q(u)(\dot{\overline{r}} \times \ddot{\overline{r}})]du \tag{1.1'}$$

where $P_i(u)$, $i = 1, 2$, $Q(u)$ are arbitary integrable functions, too. □

From (1.3) and Definition 0.1 we have

Corollary 1.1. *A curve C_ε being infinitesimal bending of a curve C is*

$$\overline{r}_\varepsilon = \overline{r}(u) + \varepsilon \int [p(u)\,\overline{n}(u) + q(u)\,\overline{b}(u)]du\,,$$

or

$$\overline{r}_\varepsilon = \overline{r}(u) + \varepsilon \int [P_1(u)\,\dot{\overline{r}} + P_2(u)\ddot{\overline{r}} + Q(u)(\dot{\overline{r}} \times \ddot{\overline{r}})]du$$

where $P_i(u)$, $i = 1, 2$, $Q(u)$ are arbitrary functions, ($\varepsilon \geq 0$, $\varepsilon \to 0$, $\varepsilon \in \mathcal{R}$). □

2. Infinitesimal bending of a circle

Let us investigate infinitesimal bending of a circle

$$\overline{r} = \cos u,\ \sin u, 0) \quad (\text{or } x^2 + y^2 = 1)\,. \tag{2.1}$$

Here is $R = 1$ and $u = s$. i.e. curve can be parametrized by arc length and we have

$$\begin{aligned} \dot{\overline{r}} = \overline{r}' = \overline{t}\,, &\quad \ddot{\overline{r}} = \overline{r}'' = K\overline{n}\,, \\ |\overline{r}'| = 1\,, &\quad |\overline{r}' \times \overline{r}''| = |\overline{r}''| = K\,, \\ \overline{n} = \frac{\overline{r}''}{K} &= R\,\overline{r}''\,. \end{aligned} \tag{2.2}$$

From (2.1) we have

$$\overline{b} = \frac{\overline{r}' \times \overline{r}''}{|\overline{r}''|} = R(\overline{r}' \times \overline{r}'')\,. \tag{2.3}$$

Since

$$\overline{r}' = (-\sin u,\ \cos u, 0), \quad \overline{r}'' = (-\cos u,\ -\sin u, 0),$$

$$\overline{r}' \times \overline{r}'' = \sin^2 u\overline{k} + \cos^2 u\overline{k} = \overline{k}\,,$$

we obtain

$$\overline{n} = 1 \cdot \overline{r}'' = (-\cos u,\ -\sin u, 0) = -\overline{r}(u)\,.$$

According to (1.1) and previous relations we get infinitesimal bending field for the circle (2.1)

$$\overline{z}(u) = \int \left[p(u)\, R(u)\overline{r}'' + q(u)\, R(u)\, \left(\overline{r}'(u) \times \overline{r}''(u)\right)\right] du$$

or

$$\overline{z}(u) = \int \left[p(u)(-\cos u\overline{i} - \sin u\overline{j}) + q(u)\overline{k}\right] du + C\,, \tag{2.4}$$

where $p(u)$, $q(u)$ are arbitrary functions.

We will not study infinitesimal bending only analytically but will do this by drawing curves. Computer program *Mathematica* [7], [3] permits us to reproduce any infinitesimal bending of a curve.

Let us consider some characteristic cases.

1) For $p(u) = C = 0, \quad q(u) = 1,$ infinitesimal bending field is

$$\overline{z}(u) = u\overline{k}. \tag{2.5}$$

Infinitesimal bending of circle C (2.1) is

$$C_\varepsilon : \overline{r}_\varepsilon = \overline{r}(u) + \varepsilon \overline{z}(u) = \overline{r}(u) + \varepsilon u \overline{k}, \tag{2.6}$$

i.e. the circle is by infinitesimal bending included in a family of helices

$$\overline{r}_\varepsilon = (\cos u, \sin u, \varepsilon u) \tag{2.7}$$

(see Fig. 1).

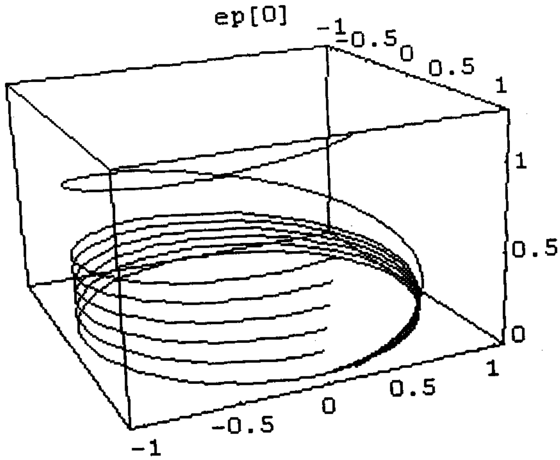

Fig. 1. Infinitesimal bending of circle 1)

Let us examine if the relation (1.4) is in force i.e. if a field (2.5) is a field of infinitesimal bending. Calculating

$$d\overline{r} \cdot d\overline{z} = (-\sin u \, du \, \overline{i} + \cos u \, du \, \overline{j}) \cdot (du \, \overline{k}) = 0$$

we confirm this. Also according to

$$ds^2 = dx^2 + dy^2 = (-\sin u \, du)^2 + (\cos u \, du)^2 = du^2$$

and

$$ds_\varepsilon^2 = ds^2 + \varepsilon^2 du^2$$

we have

$$ds_\varepsilon^2 - ds^2 = \varepsilon^2 du^2 = o(\varepsilon),$$

i.e. we have (0.3).

For the circle (2.1) curvature is $K = \dfrac{1}{R} = 1$, and torsion $r = 0$ (plane curve). For deformed circle (2.7) is

$$K_\varepsilon = \frac{1}{1+\varepsilon^2}, \quad \tau_\varepsilon = \frac{\varepsilon}{1+\varepsilon^2}.$$

Variation of curvature of a circle under infinitesimal bending is

$$\delta K = \frac{\partial K_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} = \frac{-2\varepsilon}{(1+\varepsilon^2)^2}|_{\varepsilon=0} = 0,$$

and variation of torsion

$$\delta\tau = \frac{\partial \tau_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0}, \quad \frac{1+\varepsilon^2-2\varepsilon^2}{(1+\varepsilon^2)^2}|_{\varepsilon=0} = 1,$$

i.e. the circle does not remain plane curve.

2) For $p(u) = C = 0$, $q(u) = 2\pi - u$ according to (2.4) we get infinitesimal bending field

$$\overline{z}(u) = u(2\pi - u)\overline{k}. \tag{2.8}$$

The curve we get under infinitesimal bending of a circle with this infinitesimal bending field is

$$C_\varepsilon: \overline{r}_\varepsilon = \cos u\overline{i} + \sin u\overline{j} + \varepsilon u(2\pi - u)\overline{k}.$$

This curve is on cylinder $x^2 + y^2 = 1$, but as $z_\varepsilon(u=0) = z_\varepsilon(u=2\pi) = 0$, the curve is closed. Thus curve is not a helix (Fig. 2). As

$$d\overline{r} \cdot d\overline{z} = 0,$$

and also

$$ds_\varepsilon^2 - ds^2 = \varepsilon^2 u^2 (2\pi - u)^2\, du^2 = o(\varepsilon)$$

a field (2.8) is a field of infinitesimal bending of the circle (2.1).

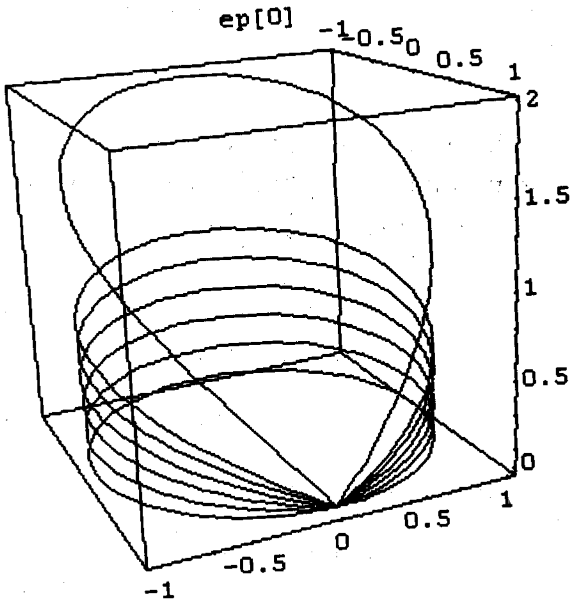

Fig. 2. Infinitesimal bending of circle 2)

Let us calculate variation of curvature and torsion. From

$$\tau_\varepsilon = \frac{2\varepsilon(\sin 2u + \pi - u)}{1 + 4\varepsilon^2(1+\pi)^2 + u^2 + 2(\pi - u)\sin 2u - \pi u},$$

$$K_\varepsilon = \frac{\sqrt{1 + 4\varepsilon^2(1 + \pi^2 + \cdots)}}{(1 + 4\pi^2\varepsilon^2 - 8\pi\varepsilon^2 u + 4\varepsilon^2 u^2)^{3/2}},$$

variation of curvature and torsion is

$$\delta\tau = \frac{\partial \tau_\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} = 2\sin 2u + \pi - u) \neq 0, \quad \delta K = 0,$$

i.e. a circle deforms to a curve that is not plane.

3) For $p(u) = q(u) = 1$ infinitesimal bending field is

$$\overline{z}(u) = -\sin u\overline{i} + \cos u\overline{j}) + u\dot{\overline{k}}.$$

Deformed curves C_ε are

$$C_\varepsilon : \overline{r}_\varepsilon = (\cos u - \varepsilon\sin u,\ \sin u + \varepsilon\cos u,\ \varepsilon u).$$

Curves C_ε are not on cylinder $x^2 + y^2 = 1$. Variation of the torsion and curvature are non-zero.

3. Infinitesimal bending of plane closed curve remaining plane curve under infinitesimal bending of this curve

We vill consider closed plane curve in polar coordinates

$$K: \rho = \rho(\theta), \quad \theta \in [0, 2\pi]. \tag{3.1}$$

Under infinitesimal bending this curve is included in family of curves

$$K_\varepsilon: \rho_\varepsilon = \rho_\varepsilon(\theta), \quad \theta \in [0, 2\pi], \quad (\varepsilon \geq 0, \quad \varepsilon \to 0). \tag{3.2}$$

Equations of these curves are in vector form:

$$K: \overline{r} = \overline{r}(\theta), \quad \theta \in [0, 2\pi], \tag{3.3}$$

$$K_\varepsilon: \overline{r}_\varepsilon = \overline{r}(\theta) + \varepsilon \overline{z}(\theta), \quad \theta \in [0, 2\pi], \tag{3.4}$$

where $\overline{z}(\theta)$ is a field of infinitesimal bending. We shall consider piecewise smooth curve. At the points where the curve is not regular we choose infinitesimal bending field continuous along a curve, i.e.

$$\overline{z}(\theta - 0) = \overline{z}(\theta + 0). \tag{3.5}$$

Theorem 3.1. *Infinitesimal bending field that plane curve*

$$K: \rho = \rho(\theta)$$

under infinitesimal bending includes in a family of plane curves

$$K_\varepsilon: \rho_\varepsilon = \rho_\varepsilon(\theta), \quad (\varepsilon \geq 0, \quad \varepsilon \to 0),$$

is

$$\overline{z}(\theta) = \rho(\theta) \sin\theta i - \rho(\theta) \cos\theta j. \tag{3.6}$$

Proof. Necessary and sufficient condition for a field $\overline{z}(\theta)$ to be infinitesimal bending field of a curve $\overline{z}(\theta)$ is

$$d\overline{r} \cdot d\overline{z} = 0 \tag{3.7}$$

i.e.

$$\dot{\overline{r}}(\theta) \cdot \dot{\overline{z}}(\theta) = 0\,. \tag{3.8}$$

Hence we conclude that $\dot{\overline{z}} \perp \dot{\overline{r}}$ i.e. that $\dot{\overline{z}}$ is in a normal plane of a curve i.e.

$$\dot{\overline{z}}(\theta) = p(\theta)\overline{n} + q(\theta)\overline{b}\,, \tag{3.9}$$

where $p(\theta)$ and $q(\theta)$ are arbitrary functions. In order to stay in the plane of the curve we choose

$$q(\theta) = 0\,. \tag{3.10}$$

So, we have

$$\dot{\overline{z}}(\theta) = p(\theta)\,\overline{n}(\theta)$$

i.e.

$$\overline{z}(\theta) = \int p(\theta)\,\overline{n}(\theta) + \overline{c}_1\,. \tag{3.11}$$

We will take $\overline{c}_1 = 0$.

As the equation of a curve K in vector form is

$$K\colon \overline{r}(\theta) = \rho(\theta)\,\cos(\theta)\overline{i} + \rho(\theta)\,\sin\theta\overline{j} \tag{3.12}$$

we calculate infinitesimal bending field of this curve according to (3.11). Substituting $\overline{n}$ with respect to (1.4) at (3.11), we have

$$\overline{z} = \int p(\theta)\,\frac{(\rho\ddot{\rho} - \rho^2 - 2\dot{\rho}^2)[(\rho\,\cos\theta + \dot{\rho}\,\sin\theta)\overline{i} + (\rho\,\sin\theta - \dot{\rho}\,\cos\theta)\overline{j}]}{\sqrt{\rho^2 + \dot{\rho}^2}\;|\rho\ddot{\rho} - \rho^2 - 2\dot{\rho}^2|}\,d\theta\,.$$

Since

$$\rho(\theta)\cos\theta + \dot{\rho}(\theta)\sin\theta = \dot{\big(\rho(\theta)\sin\theta\big)};\;\; \rho(\theta)\sin\theta - \dot{\rho}(\theta)\cos\theta = -\dot{\big(\rho(\theta)\cos\theta\big)};$$

we choose

$$p(\theta) = |\dot{\overline{r}}| = \sqrt{\rho(\theta)^2 + \dot{\rho}(\theta)^2}\,,$$

that gives

$$\begin{aligned}\overline{z}(\theta) &= \int \big[\dot{\big(\rho(\theta)\,\sin\theta\big)}\overline{i} - \dot{\big(\rho(\theta)\,\cos\theta\big)}\overline{j}\big]d\theta = \\ &= \dot{\rho}(\theta)\,\sin\theta\overline{i} - \rho(\theta)\,\cos\theta\overline{j}\,.\end{aligned} \tag{3.13}$$

So we have determined infinitesimal bending field that plane curve includes in a family of plane curves under infinitesimal bending. □

Corollary 3.1. *If the plane curve*

$$K\colon \rho = \rho(\theta)\,, \quad \theta \in [0, 2\pi]\,,$$

under infinitesimal bending stays at original plane, the equation of deformed curve will be

$$K_\varepsilon\colon \rho_\varepsilon(\theta) = \rho(\theta)\sqrt{1+\varepsilon^2}\,. \tag{3.14}$$

Proof. According to previous theorem we have

$$\overline{r}_\varepsilon = \overline{r} + \varepsilon\overline{z} = \rho(\theta)\,(\cos\theta + \varepsilon\,\sin\theta)\overline{i} + \rho(\theta)(\sin\theta - \varepsilon\,\cos\theta)\overline{j}$$

i.e.

$$\rho_\varepsilon(\theta) = \rho(\theta)\sqrt{1+\varepsilon^2}\,, \quad \Box \tag{3.15}$$

Example 3.1. Infinitesimal bending field of the circle $\rho(\theta) = a$, $\theta \in [0, 2\pi]$ i.e.

$$K\colon \overline{r}(\theta) = a\,\cos\theta\overline{i} + a\,\sin\theta\overline{j}\,,$$

is

$$\overline{z}(\theta) = \rho(\theta)\,\sin\theta\overline{i} - \rho(\theta)\,\cos\theta\overline{j} = a(\sin\theta\overline{i} - \cos\theta\overline{j})\,.$$

A family of curves that is infinitesimal bending of the circle K is

$$K_\varepsilon\colon \overline{r}_\varepsilon(\theta) = \overline{r}(\theta) + \varepsilon\overline{z}(\theta) = a(\cos\theta + \varepsilon\,\sin\theta)\overline{i} + a(\sin\theta - \varepsilon\,\cos\theta)\overline{j}\,.$$

The curves K_ε are concentric, homothetic circles, because

$$\rho_\varepsilon = \rho\sqrt{1+\varepsilon^2}\,, \quad \theta \in [0.2\pi]\,.$$

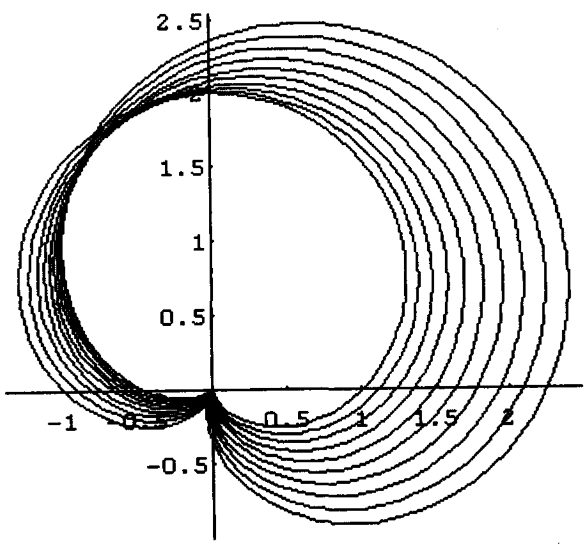

Fig. 3. **Infinitesimal bending of cardioid**

Example 3.2. For cardioid

$$\rho(\theta) = 1 + \sin\theta\,, \quad \theta \in [0, 2\pi]\,,$$

i.e.

$$K: \overline{r} = (1 + \sin\theta)\cos\theta\overline{i} + (1 + \sin\theta)\sin\theta\overline{j}$$

infinitesimal bending field is

$$\overline{z}(\theta) = (1 + \sin\theta)\sin\theta\overline{i} - (1 + \sin\theta)\cos\theta\overline{j}\,.$$

Curves that are infinitesimal bending of the cardioid K are (Fig. 3.)

$$K_\varepsilon: \overline{r} = (1 + \sin\theta)(\cos\theta + \varepsilon\,\sin\theta)\overline{i} + (1 + \sin\theta)(\sin\theta - \varepsilon\,\cos\theta)\overline{j}\,,$$

i.e. we have

$$\rho_\varepsilon = (1 + \sin\theta)\sqrt{1 + \varepsilon^2}\,, \quad \theta \in [0, 2\pi]\,.$$

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ИНФИНИТЕЗИМАЛНА ДЕФОРМАЦИЈА НА КРИВИ

Љубица Велимировиќ

Резиме

Во оваа работа се разгледува инфинитезимална деформација на криви во $\mathcal{R}^3$. Дадено е полето на инфинитезимална деформација и деформираната крива. Истражувана е инфинитезимална деформација на кружница. Разгледувана е варијација на торзијата и кривината на кружницата при инфинитезималната деформација. Дадена е инфинитезималната деформација која рамнинската крива ја деформира во рамнинска крива. Инфинитезималната деформација е проучувана не само аналитички, туку и со исцртување.

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